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HARMONIC ANALYSIS ON TOTALLY DISCONNECTED LOCALLY COMPACT GROUPS

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Preface

This thesis marks the culmination of my doctoral studies, which has been a long and rewarding experience, and I would like to briefly acknowledge those who have helped make it possible.

My deepest thanks go to my supervisors, Pierre-Emmanuel Caprace and Timothée Marquis, for their careful guidance, expertise, and support over the past three years. I must thank them, along with Tom De Medts, for employing me on their joint Excellence of Science grant, which has more than generously supported my studies and travel over the course of my doctorate. I greatly appreciate their flexibility in allowing me to explore my own research interests, which deviated largely from the initial goals of their grant.

Throughout my PhD, I have had the privilege to travel to many conferences and workshops, and visit a number of departments. I would like to thank my collaborators, peers, and senior researchers whom I have met during my PhD. Our informal discussions have enriched my work and mathematical development.

Finally, I would like to thank my girlfriend and my parents for their continuous support and encouragement behind the scenes.

Abstract

The harmonic analysis of locally compact groups has been studied for over 100 years, and has been investigated deeply, particularly in the settings of abelian groups, compact groups, and connected Lie groups. While many harmonic analysis properties are well understood for these classes of groups, in contrast, our understanding of them remains significantly underdeveloped in the setting of totally disconnected locally compact (tdlc) groups.

This thesis studies harmonic analysis properties coming from unitary representation theory (e.g. the type I and CCR properties), and from Banach algebra theory (e.g. the Wiener, Hermitian and $*$-regular properties), for many classes of tdlc groups. The following questions underpin the work carried out in this thesis:

(i) Do the established results regarding these properties for connected Lie groups have direct analogues for tdlc groups?
(ii) How does the dichotomy between amenable and non-amenable groups influence these properties in the tdlc setting?

The thesis starts in Chapter 1 with an introduction, where the above listed properties are linked to a central topic, spectral synthesis, which is rooted in classical questions from Fourier analysis.

Chapters 2, 3 and 5 investigate Banach algebras on tdlc groups, and in particular, study the Hermitian, Wiener and $*$-regular properties. It is known that the Banach $*$-algebra $L^1(G)$ cannot be Hermitian or $*$-regular for non-amenable groups, and in Chapter 2, we further show that for many non-amenable tdlc groups, $L^1(G)$ is also not Wiener. In Chapter 3, we study locally elliptic groups, which are groups that are ascending unions of compact open subgroups, and hence amenable. For such groups it is known that $L^1(G)$ is Wiener and $*$-regular whenever it is Hermitian. In Chapter 3, we show that this holds for a much broader class of Banach $*$-algebras on locally elliptic groups which are $*$-subalgebras of $L^1(G)$, and develop numerous criterions for determining when these algebras are Hermitian. In Chapter 5, we show that $L^1(G)$ is Hermitian when G is the affine group of a (non-archimedean) local field, resolving a long standing open question.

In Chapter 4, we investigate the unitary representation theory of two-step nilpotent locally compact groups, with a focus on tdlc groups and the type I and CCR properties. While Chapters 2, 3 and 5 on Banach algebras show that the analogous results for Lie groups hold almost identically for tdlc groups, Chapter 4 provides some contrasting phenomena. On one hand, we tackle the question of whether unipotent groups over positive characteristic local fields are type I and CCR, and provide positive results in the two-step nilpotent case. On the other hand, we produce uncountably many examples of non-type I p-torsion two-step nilpotent tdlc groups, providing contrasting phenomena to the setting of connected Lie groups where all nilpotent groups are CCR (and hence type I).

Contents

CHAPTER 1

Introduction

1.1. Introduction and motivation to studying harmonic analysis on totally disconnected locally compact groups

Fourier analysis is, at its core, the study of how a function can be broken down into "simpler" components. The classical view point was to study functions on abelian groups such as $\mathbb{R}^n$, $\mathbb{Z}^n$ and $\mathbb{T}^n$, and decomposes them into sums or integrals involving sine and cosine functions. This idea can be traced all the way back to the work of Joseph Fourier in 1822 [**54**], where he showed that these techniques can be used to study heat transfer in physics. Since its invention, Fourier analysis has been a centre point of modern mathematics, and it has had an incredible amount of applications in physics, engineering, theoretical mathematics, and more broadly, science.

During the 20th century, the point of view of abstract harmonic analysis was to generalise these concepts of Fourier analysis from the 19th and early 20th centuries to a general locally compact group G. This was motivated to a large extent by the rapid development of quantum mechanics during the 1920's and 1930's, which stressed the importance of understanding the harmonic analysis and unitary representations of locally compact groups, and in particular, Lie groups [**101**]. The harmonic analysis of abelian groups and compact groups was investigated deeply during the 1920/30's and is very well understood [**116, 121, 52**]. Similarly, the harmonic analysis of connected Lie groups has been studied extensively throughout the 20th century and is well developed in the majority of cases [**55, 146, 147**].

Now, if G is an arbitrary locally compact group, then the connected component of the identity in G, denoted G°, is a closed normal connected subgroup of G, and the quotient G/G° is a totally disconnected locally compact group (to be abbreviated ***tdlc group*** from now on). Thus every locally compact group is an extension of a connected group by a tdlc group. Moreover, by the solution to Hilbert's fifth problem, the connected locally compact group G° can be "approximated" by connected Lie groups in a certain sense [**104**]. Thus, understanding the harmonic analysis of connected locally compact groups goes hand-in-hand with studying the harmonic analysis of connected Lie groups, which is well developed.

On the other hand, the harmonic analysis of tdlc groups, in full generality, is not well developed. One issue here is that the class of tdlc groups contains all discrete groups, and hence all groups, so saying something about the harmonic analysis of all tdlc groups is unrealistic. This is further complicated by the fact that any discrete group G which is not virtually abelian is not "type I", which effectively means that it is impossible to describe all the irreducible unitary representations of G, which is fundamental to studying the harmonic analysis of G. However, it is a reasonable goal to pursue a deeper theory of the harmonic analysis of ***non-discrete*** tdlc groups (after disregarding some degenerate cases, such as being compact-by-discrete). There are certain cases where harmonic analysis has been studied extensively in the setting of non-discrete tdlc groups, such as for Lie groups over non-discrete tdlc fields [**63, 137, 68**] and some very specific classes of groups acting on trees [**51, 34**]. However, the harmonic analysis of general non-discrete tdlc groups is still not well understood. Also, in contrast to connected groups, non-discrete tdlc groups cannot be approximated by Lie groups over tdlc fields and so their harmonic analysis must be studied extensively beyond these cases.

The 21st century has seen significant progress made on the structure theory of non-discrete tdlc groups [**24, 25**]. This progress was initiated by Willis' influential work from the 1990's regarding the scale theory of tdlc groups [**151**] which has now formed the back bone of modern work in tdlc group theory. Furthermore, viewing tdlc groups as automorphism groups of discrete structures, such as automorphism groups of graphs and simplicial complexes [**78, 2**], has also formed a major part of the modern theory of tdlc groups and played an important role in recent advances. Notable points of progress this century include the development of chief series of compactly generated tdlc groups [**129**], the structure theory of contraction groups [**58, 59, 60**], and the classifications of (P)-closed groups and boundary 2-transitive groups acting on trees [**125, 128**].

This recent progress in the structure theory of non-discrete tdlc groups enables one to now pursue a deeper theory of harmonic analysis on such groups. This is supported by an influx of new papers on this topic in the past 5-10 years (see for example [**67, 23, 134, 135**]), including papers by myself and collaborators [**27, 28, 29, 22, 30**], some of which make up the contents of this thesis.

1.2. Fourier analysis on locally compact groups

Let G be a locally compact group. It is well known that G possesses a Radon measure μ which is left-invariant under translations by G. The measure μ is unique up to scalar multiples and is called the ***Haar measure*** on G (see [**52**, §2.2] and [**62**]). The existence of this measure enables one to do integration and analysis on

G, and, over the past 100 years, has led to an extensive theory of Fourier/harmonic analysis on locally compact groups.

The existence of the Haar measure also allows one to study spaces of functions on G defined by various integrability conditions. The most common is the so called ***L^1-group algebra***, denoted by $L^1(G)$, consisting of all measurable functions $f : G \to \mathbb{C}$ which are absolutely integrable i.e. such that

$$\|f\|_{L^1(G)} := \int_G |f(x)| \, \mathrm{d}\mu(x) < \infty.$$

The space $L^1(G)$ is a Banach space with the above norm. It is also a ***Banach *-algebra*** when equipped with the convolution product

$$(f * g)(x) := \int_G f(y)g(y^{-1}x) \, \mathrm{d}\mu(x)$$

and conjugate linear norm preserving involution

$$f^*(x) := \Delta(x^{-1})\overline{f(x^{-1})}$$

where Δ is the so called ***modular function*** on G [**52**, §2.4]. The involvement of the modular function Δ in the definition of this involution is solely for the purpose of making it norm preserving, and for many classes of locally compact groups, including many studied in this thesis, Δ is identically 1 and hence can be disregarded from the definition.

The Fourier transform on the locally compact group G can then be viewed as a transform acting on functions in $L^1(G)$, which we will now work towards defining. To do this, one must first have at least a basic understanding of unitary group representations which we will now discuss briefly. Recall that, given a Hilbert space $\mathcal{H}$, the set of all unitary operators on $\mathcal{H}$, denoted $\mathcal{U}(\mathcal{H})$, is a group under composition. It is in fact a topological group when equipped with the topology of pointwise convergence (called the strong operator topology in this case) and is a locally compact group if and only if $\mathcal{H}$ is finite dimensional [**150**, Example 1.6 & 1.28]. A ***unitary representation*** of G is then a pair $(\pi, \mathcal{H}_\pi)$, where $\mathcal{H}_\pi$ is a Hilbert space, and $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a continuous group homomorphism. The representation $(\pi, \mathcal{H}_\pi)$ is called ***(topologically) irreducible*** if there are no proper non-trivial closed subspaces in $\mathcal{H}_\pi$ which are invariant under $\pi(G)$. Two unitary representations $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ of G are called ***equivalent*** if there exists a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ such that $U\pi(g)U^{-1} = \sigma(g)$ for all $g \in G$. The collection of all equivalence classes of irreducible unitary representations of G forms a set, denoted $\widehat{G}$, called the ***unitary dual*** of G. The set $\widehat{G}$ is a topological space when equipped with a canonical topology, called the ***Fell topology*** [**7**, §1.C], and a measurable space when equipped with a canonical measure structure, called the ***Mackey-Borel structure*** [**42**, §3.8 & §18.5].

In the case that G is an abelian group, one can say a lot more about the structures in the above paragraph. Indeed, if G is abelian, then the irreducible unitary representations are all 1-dimensional by Schur's lemma, and are identified with the ***(unitary) characters*** of G i.e. continuous homomorphisms $\chi : G \to \mathbb{T}$. The set $\widehat{G}$ then becomes a locally compact abelian group with operation pointwise multiplication of characters, and the Fell topology is precisely the compact-open topology on $\widehat{G}$. In contrast, when G is a non-abelian group, the unitary dual has no canonical group structure.

To further motivate the definition of the Fourier transform on a locally compact group, we provide below a description of the unitary dual of some classical examples of abelian groups.

Example 1.1. (i) $\widehat{\mathbb{R}} = \{\chi_y : y \in \mathbb{R}\}$ where $\chi_y(x) := e^{2\pi i x y}$ for all $x \in \mathbb{R}$;
(ii) $\widehat{\mathbb{Z}} = \{\chi_\theta : \theta \in [0,1)\}$ where $\chi_\theta(n) := e^{2\pi i n\theta}$ for all $n \in \mathbb{Z}$;
(iii) $\widehat{\mathbb{T}} = \{\chi_n : n \in \mathbb{Z}\}$ where $\chi_n(\theta) := e^{2\pi i n\theta}$ for all $\theta \in \mathbb{T} \cong [0,1)$.

We now give the definition of the Fourier transform on an arbitrary locally compact group G. The Fourier transform acts on $L^1(G)$ as follows: given $f \in L^1(G)$, the Fourier transform of f, denoted $\hat{f}$, is a function on the unitary dual $\widehat{G}$ of G and is given by the following operator valued integral

$$\hat{f}(\pi) := \int_G f(x)\pi(x)\,\mathrm{d}\mu(x)$$

for $\pi \in \widehat{G}$ (we identify an equivalence class in $\widehat{G}$ with one of its representatives). The expression $\hat{f}(\pi)$ is an element of $\mathcal{B}(\mathcal{H}_\pi)$, the space of bounded operators on the Hilbert space $\mathcal{H}_\pi$. The action of $\hat{f}(\pi)$ on $\mathcal{H}_\pi$ can be defined in terms of its action on inner products in $\mathcal{H}_\pi$, which is given by

$$\langle \hat{f}(\pi)\xi, \zeta\rangle_{\mathcal{H}_\pi} := \int_G f(x)\langle \pi(x)\xi, \zeta\rangle_{\mathcal{H}_\pi}\ \mathrm{d}\mu(x)$$

for $\xi, \zeta \in \mathcal{H}_\pi$. Of course, in the case that G is an abelian group, then $\pi \in \widehat{G}$ is a character of G, and hence $\hat{f}(\pi)$ is an integral of complex valued functions. In the case that G is one of the groups $\mathbb{R}, \mathbb{Z}, \mathbb{T}$ or similar, then $\hat{f}(\pi)$ becomes an integral of f against some exponential functions on the given group, as shown by the above example. This will then give the usual Fourier transform studied in undergraduate analysis.

One sees from this construction of the Fourier transform on $L^1(G)$ that, to understand Fourier/harmonic analysis on the group G, one would at least have to understand, firstly, the set $\widehat{G}$ and hence the unitary representation theory of G, and secondly, the structure of $L^1(G)$. This is the main perspective of this thesis: to initiate further investigation into these ideas for tdlc groups.

1.3. Spectral synthesis

The topic of spectral synthesis concerns understanding the (ideal) structure of $L^1(G)$ for general locally compact groups G. Here we will give a brief introduction to this topic and recent research in this area which will motivate a large portion of the thesis. In the following, G will be an arbitrary locally compact group and an ideal in $L^1(G)$ is always understood to be two-sided.

Given a closed ideal $I \subseteq L^1(G)$, one defines the following subset of $\widehat{G}$, called the ***hull*** of I:

$$h(I) := \{\pi \in \widehat{G} : \hat{f}(\pi) = 0 \,\forall f \in I\}.$$

Similarly, given $S \subseteq \widehat{G}$ closed, one defines a closed ideal in $L^1(G)$, called the ***kernel*** of S:

$$k(S) := \{f \in L^1(G) : \hat{f}(\pi) = 0 \,\forall \pi \in S\}.$$

It can be shown that $k(S)$ is the largest closed ideal $I \subseteq L^1(G)$ with $h(I) = S$. The set $S \subseteq \widehat{G}$ is called a ***set of synthesis*** if $k(S)$ is the only closed ideal in $L^1(G)$ with hull equal to S. The topic of ***spectral synthesis*** can be viewed in one way as determining which sets in the unitary dual of a locally compact group are sets of synthesis. However, the term of 'spectral synthesis' can be used much more broadly than this to refer to an array of different analytical questions on various function spaces; see for example [**10**].

The topic of spectral synthesis is an extremely interesting and heavily studied topic, particularly in the case of abelian groups (see [**65**, Chapter 10], [**130**, Chapter 7] and [**74**, Chapter 6]). It is well known that if G is a compact abelian group, then every subset of $\widehat{G}$ is set of synthesis ([**65**, §42] and [**130**, §7.6]). This means that, for a compact abelian group G, the closed ideals in $L^1(G)$ are uniquely determined by their hull. On the other hand, if G is now a non-compact locally compact abelian group, then there always exists a closed subset of $\widehat{G}$ which is not a set of synthesis (see [**65**, §42] and [**130**, §7.6] again). In particular, closed ideals in $L^1(G)$ for a non-compact abelian group are not always determined by their hulls. In this direction, it is a classical result of Schwartz [**132**] that the unit sphere in $\mathbb{R}^d$ ($d \geqslant 3$) is not a set of synthesis, where you view $\mathbb{R}^d$ as its own unitary dual.

In the non-abelian setting, there are also a number of interesting results regarding spectral synthesis. A large portion of these results are concentrated in the setting of compactly generated groups with polynomial growth [**50, 45**], and in particular, connected nilpotent real Lie groups [**98, 99, 94, 9**]. Associated to every connected nilpotent real Lie group N is a geometric space, called the ***coadjoint orbit space*** [**76, 77**], and denoted by $\mathcal{O}(N)$. The space $\mathcal{O}(N)$ is the space of orbits with respect to the action of N on the vector space dual of its Lie algebra. It is a well known result, called the ***Kirillov orbit method***, that $\widehat{N}$ is homeomorphic to $\mathcal{O}(N)$

[**21**]. It is shown in [**99**, Theorem 4.10(ii)] that if a point $\pi \in \widehat{N}$ corresponds to a hyperplane in $\mathcal{O}(N)$, then the set $\{\pi\}$ is a set of synthesis. In the case that N is two-step nilpotent, then all the points in $\widehat{N}$ are sets of synthesis [**98**]. However, there exist three-step nilpotent connected real Lie groups which have a point in the dual which is not a set of synthesis [**98**].

To complete this section, we will give some motivation for why compactly generated groups of polynomial growth are of interest here, and in doing so, introduce the idea of studying minimal ideals in $L^1(G)$ of a given hull. Suppose that G is a locally compact abelian group and let $S \subseteq \widehat{G}$ be closed. Then define the following set:

$$j(S) := \{f \in L^1(G) : \operatorname{supp}(\hat{f}) \text{ is compact and disjoint from } S\}.$$

It can be shown that $j(S)$ is an ideal in $L^1(G)$, and its closure, $\overline{j(S)}$, is the smallest closed ideal in $L^1(G)$ with hull equal to S [**130**, §7.2.5]. In particular, if an ideal $I \subseteq L^1(G)$ satisfies $h(I) = S$, then $j(S) \subseteq I \subseteq k(S)$. If S is a set of synthesis then we of course have that $\overline{j(S)} = k(S)$. If S is not a set of synthesis, then to understand the closed ideals in $L^1(G)$ with hull equal to S, the problem reduces to understanding the ideal theory of the quotient algebra $k(S)/\overline{j(S)}$.

Unfortunately, for general locally compact groups G, the algebra $L^1(G)$ does not always have a minimal ideal for each hull which adds a degree of difficulty to understanding ideals corresponding to sets which are not sets of synthesis. However, locally compact groups of polynomial growth are a class of groups where there *can* exist a minimal ideal in $L^1(G)$ for each hull. To explain this further, one calls a locally compact group G ***Hermitian*** if for all self adjoint $f = f^* \in L^1(G)$, the spectrum of f in $L^1(G)$, denoted $\sigma_{L^1(G)}(f)$, is contained in $\mathbb{R}$. It should be noted that, for example, every nilpotent group and every compactly generated group of polynomial growth is Hermitian [**95, 92**]. Furthermore, it was recently shown that Hermitian groups are amenable [**131**]. It is then an important result of Ludwig [**96**, Theorem 1] that if G is a Hermitian locally compact group with polynomial growth, then corresponding to every closed subset $S \subseteq \widehat{G}$, there exists an ideal $j(S) \subseteq L^1(G)$ such that $\overline{j(S)}$ is the smallest closed ideal in $L^1(G)$ with hull equal to S. Thus for Hermitian groups with polynomial growth, one can pursue a deeper understanding of the structure of the algebras $k(S)/\overline{j(S)}$ for each closed $S \subseteq \widehat{G}$. It is shown, for example, that for a connected nilpotent Lie group N, the algebras $k(S)/\overline{j(S)}$ are always nilpotent, and their ideals have been computed when N is three-step nilpotent and S is a singleton set [**97**].

1.4. Wiener's Tauberian theorem

As mentioned in the previous section, if G is an abelian group, then as soon as it is non-compact, there exists a closed subset in $\widehat{G}$ which is not a set of synthesis. Even in the case of $\mathbb{R}^d$, there are natural sets in its unitary dual, such as spheres, which are not sets of synthesis. It is, on the other hand, a well known theorem of Norbert Wiener that the empty set $\varnothing \subset \mathbb{R}^d$ is a set of synthesis, where we consider $\mathbb{R}^d$ as its own unitary dual [**149**]. This result is typically referred to as ***Wiener's Tauberian theorem*** and it has many other equivalent/similar formulations (see [**149**], [**130**, §7.2] and [**115**, §12.5.21]). More generally, one can ask, given an arbitrary locally compact group G, whether $\varnothing \subset \widehat{G}$ is a set of synthesis. In the case that $\varnothing \subset \widehat{G}$ is a set of synthesis, then we call G a ***Wiener group***. The property of a group G being Wiener is equivalent to the property that, for every proper closed ideal $I \subset L^1(G)$, the hull of I, $h(I)$, is non-empty.

The question of understanding which locally compact groups are Wiener was a big question in Banach algebra theory during the mid-to-late 20th century. During this time a large focus of the study of Wiener groups was concentrated on understanding this property for connected locally compact groups and discrete groups. There has been, however, some lack of understanding about this property for non-discrete tdlc groups.

It is known that all abelian groups, compact groups, and nilpotent groups are Wiener (see [**130**, §7.2], [**92**] and [**95**]). Furthermore, all non-Wiener connected exponential Lie groups have been computed up to real dimensional 6 [**118**], and the lowest dimensional example occurs in dimension 4 (the so called "Poguntke group" [**89**]). More recently, in the early 21st century, Losert resolved a long standing open conjecture, showing that every compactly generated group of polynomial growth is Hermitian and hence Wiener [**92**]. This result includes many earlier examples of discrete groups that were known to be Wiener. Other than some examples of exponential Lie groups, the only other known class of non-Wiener groups are non-compact connected semisimple Lie groups [**86**, Appendix]. It should be noted that all the known examples of Wiener groups are amenable, and it is an open question to determine whether Wiener groups are amenable. Furthermore, the Wiener property is closely linked with the Hermitian property in many situations: for "weakly Wiener" groups (see [**115**, Definition 12.5.21] and [**88**, Definition 4]), which includes all discrete groups and all groups with polynomial growth, Hermitian implies Wiener. More information about Wiener and weakly Wiener groups can be found in [**115**, §11.5 & §12.6.36] and [**88**].

1.5. Type I groups

In the previous sections we have described the interest in understanding the ideal theory and structure of the Banach algebra $L^1(G)$ and its relations to studying harmonic analysis on G. As has been seen, a critical part of understanding the ideal theory of $L^1(G)$ is utilising a correspondence between the closed ideals of

$L^1(G)$ and the closed subsets of the unitary dual $\widehat{G}$ of G. Thus, to have a strong understanding of the ideal theory of $L^1(G)$, one would typically need to have a strong understanding of the unitary dual $\widehat{G}$ and its topology. Unfortunately, for some classes of groups, the unitary dual $\widehat{G}$ is such a large and complicated space that it is essentially impossible to describe all the irreducible unitary representations of G, nor to understand the measurable/topological structure of $\widehat{G}$. Thus it is an important question to determine, for which locally compact groups G, is it possible to get a handle of the space $\widehat{G}$ and its topology, and such groups are typically those which fall into the class of ***type I*** groups. In this section, we will give a brief introduction to the study of type I groups.

To introduce the type I property, we need to recall some basic facts on von Neumann algebras. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the space of bounded operators on $\mathcal{H}$. The space $\mathcal{B}(\mathcal{H})$ is an algebra with respect to composition of operators, and the operation of taking the adjoint of an operator is a conjugate linear norm preserving involution on $\mathcal{B}(\mathcal{H})$. In particular, $\mathcal{B}(\mathcal{H})$ is a $*$-algebra, and it in fact satisfies a stronger property, that of being a C^*-algebra [**106**, Example 2.1.3]. A ***von Neumann algebra*** is then a subalgebra of $\mathcal{B}(\mathcal{H})$ which satisfies the following properties:

(i) it is closed under taking the adjoint operation;
(ii) it contains the identity operator from $\mathcal{B}(\mathcal{H})$;
(iii) it is closed in the strong operator topology i.e. the topology of point-wise convergence.

In particular, a von Neumann algebra is a unital $*$-algebra contained in $\mathcal{B}(\mathcal{H})$ which is closed in the strong operator topology. Equivalently, one can define a von Neumann algebra as follows: given a subset $M \subseteq \mathcal{B}(\mathcal{H})$, define its ***commutant*** by

$$M' := \{T \in \mathcal{B}(\mathcal{H}) : \forall S \in M, TS = ST\}.$$

It is a classical result of von Neumann, known as ***von Neumann's double commutant theorem***, that a unital $*$-algebra $A \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $A = A''$ [**139**, Chapter II, §3]. In particular, one checks that for any subset $M \subseteq \mathcal{B}(\mathcal{H})$, the set M'' is in fact a von Neumann algebra containing M, called the ***von Neumann algebra generated by*** M.

A von Neumann algebra $A \subseteq \mathcal{B}(\mathcal{H})$ is called a ***factor*** if its centre, $Z(A) := A \cap A'$, is equal to $\mathbb{C}I_{\mathcal{B}(\mathcal{H})}$, where $I_{\mathcal{B}(\mathcal{H})}$ denotes the identity operator in $\mathcal{B}(\mathcal{H})$. In some sense, the factor von Neumann algebras are the building blocks of general von Neumann algebras: every von Neumann algebra decomposes as a "direct integral" of factor von Neumann algebras [**139**, Chapter IV, Theorem 8.21]. Here, a direct integral is some kind of measurable generalisation of a direct sum. The factor von Neumann algebras can be further classified into different types: type I_n $(n \in \mathbb{N} \cup \{\infty\})$, type

II_1, type II_∞, and type III_λ ($\lambda \in [0,1]$) [**139**, Chapter V]. The type I factor von Neumann algebras have a very simple structure: they are precisely the algebra $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The n in type I_n just corresponds to the dimension of $\mathcal{H}$. On the other hand, factor von Neumann algebras of type II and type III have a very complicated structure, and it is generally difficult to construct such algebras. Algebras of these types arise, for example, from complicated ergodic group actions [**140**, Chapter XIII].

Now let G be a locally compact group and $(\pi, \mathcal{H}_\pi)$ a unitary representation of G. Associated to π is the von Neumann algebra that it generates, which is precisely the von Neumann algebra $\pi(G)''$. Equivalently, it is the closure in the strong operator topology of the linear span of $\pi(G)$ in $\mathcal{B}(\mathcal{H}_\pi)$. The representation π is called a ***factor*** if $\pi(G)''$ is a factor von Neumann algebra. As is the case with von Neumann algebras, any unitary representation decomposes as a "direct integral" of factor unitary representations in a unique way [**42**, Chapter 8]. So every unitary representation of G can be broken down into factor representations. The locally compact group G is called ***type I*** if every factor representation of G generates a type I von Neumann algebra.

One can think of type I groups as those groups with "nice" or "tame" unitary representation theory. This is further compounded by a classical theorem of Glimm, referred to as ***Glimm's theorem*** [**57**], which shows that the property of being type I is equivalent to many other properties, including:

(i) The space $\widehat{G}$ is a standard Borel space with the Mackey-Borel structure i.e. it is Borel isomorphic to a complete separable metric space;
(ii) The Fell topology on $\widehat{G}$ is T_0-separable;
(iii) Every unitary representation of G decomposes *uniquely* as a direct integral of irreducible unitary representations.

Consequently, when G is a locally compact group which is not type I, a number of bad things happen. Firstly, there always exists a unitary representation of G which decomposes into irreducible unitary representations in two very different ways. In particular, the way in which a unitary representation decomposes into irreducibles is not an invariant of the representation for non-type I groups. Also, by Glimm's theorem, when G is not type I, the measure/topological structure of $\widehat{G}$ is so complicated that it is effectively impossible to completely describe the unitary dual of G. Thus it is an important question, not only in representation theory, but also more broadly harmonic analysis, to understand which groups are type I.

Interestingly, the property of a group being type I is implied by a stronger condition, called the CCR property, which is a property about the Fourier transform on G. Indeed, a locally compact group G is called ***CCR*** if for all $(\pi, \mathcal{H}_\pi) \in \widehat{G}$ and $f \in$

$L^1(G)$, the operator $\hat{f}(\pi) \in \mathcal{B}(\mathcal{H}_\pi)$ is compact [**52**, pg. 230]. The CCR property is equivalent to the condition that the Fell topology on $\widehat{G}$ is T_1-separable i.e. points in $\widehat{G}$ are closed [**52**, Theorem 7.7], so it implies that G is type I by Glimm's theorem.

The question of which groups are type I has been studied extensively in the literature over the past century and a lot is understood. For abelian groups and compact groups, because all of their irreducible unitary representations are finite-dimensional, it follows immediately from the definition of the Fourier transform that these groups are CCR and hence type I. These questions are also generally well understood in the setting of connected groups. For example, every connected nilpotent and connected semisimple Lie group is known to be CCR [**7**, Theorem 6.E.19]. For solvable Lie groups, however, it is a technical problem and there exist solvable Lie groups which are not type I [**3**]. Furthermore, a connected locally compact group is type I if and only if all its Lie quotients are type I (follows from [**7**, Theorem 6.E.21]), so in the connected setting, the type I property is really a question about Lie groups.

The question of which tdlc groups are type I has also been investigated extensively, however, the state of the research is still very far from having a complete understanding of the situation. Previous work in this setting has focused primarily on algebraic groups over tdlc fields and groups acting on trees. In the setting of algebraic groups, reductive groups over arbitrary local fields and linear algebraic groups over characteristic zero local fields are type I [**7**, Theorem 6.E.19]. In the setting of groups acting on trees, the "type I conjecture" has formed a centre point of modern research in the area: a non-amenable group acting minimally on a regular tree is conjectured to be type I if and only if it acts transitively on the boundary of the tree [**67, 23**]. The "only if" direction is known to be true by [**23**] but the "if" direction is still open. There is a similar conjecture for the CCR condition [**107**] which is also still open. The "if" direction of these conjectures is known to be true for non-amenable "(P)-closed" groups acting trees, where an explicit computation of their irreducible unitary representation can be produced when they act transitively on the boundary [**35, 135**]. Recent research in the area has also investigated the type I property more broadly for hyperbolic (tdlc) groups in [**23**].

1.6. Totally disconnected locally compact groups

The goal of this thesis is to progress the understanding of the harmonic analysis on locally compact groups. Given a locally compact group G, the connected component of the identity in G, denoted G°, is a closed normal connected locally compact group, and the quotient G/G° is a tdlc group. In particular, G admits the following short exact sequence

$$\{1\} \to G^\circ \to G \to G/G^\circ \to \{1\}.$$

As already mentioned in Section 1.1, every connected locally compact group can be approximated by connected Lie groups. In particular, the following holds: there exists in G° arbitrarily small compact normal subgroups such that the corresponding quotients are connected Lie groups. Many questions about connected locally compact groups, including questions coming from harmonic analysis, can be understood by passing to the Lie quotients of the given group. Since the harmonic analysis of connected Lie groups has been studied extensively in the past century, we generally have a good handle of the situation in regards to the harmonic analysis on the group G°. Thus it makes sense to try to now pursue a deeper understanding of the group G/G° and its harmonic analysis, which is the point of this thesis.

We will now give some details about tdlc groups and their structure, which is central to understanding the harmonic analysis of tdlc groups and the work in this thesis. Firstly, every tdlc group is a direct limit of its compactly generated open subgroups. Thus, it makes sense to focus on understanding the compactly generated tdlc groups at least initially, which we will from here on abbreviate as cgtdlc groups. Now let G be a cgtdlc group, let $K \subseteq G$ be a compact generating set, and choose a compact open subgroup $U \subseteq G$. Such a U always exists since, by a classical result, every tdlc group has a basis of neighbourhoods of the identity made up of compact open subgroups [**144**]. Associated to every such pair (K, U) is a so called ***Cayley-Abels graph*** of G [**78**]. This graph, which we will denote by X, is defined as follows: its set of vertices, VX, is the coset space G/U, and the set of edges is $EX := \{\{gU, hU\} : g, h \in G, \exists k \in K, gU = khU\}$. Of course, the definition of X depends on the choice of K and U, but X is unique up to quasi-isometry by the Švarc-Milnor lemma, so X is a quasi-isometric invariant of G (see also [**78**]). One notes that G acts on X vertex transitively and such that, for every vertex $v \in VX$, $G_v := \{g \in G : gv = v\}$ is a compact open subgroup of G. In particular, every cgtdlc group acts vertex transitively on a connected locally finite graph with compact open vertex stabilisers, and general tdlc groups are direct limits of such groups. Thus understanding the harmonic analysis of groups acting on graphs is critical in the study of harmonic analysis on tdlc groups.

In analogue with how parabolic and unipotent groups are fundamental in the study of reductive algebraic groups and their harmonic analysis, there are also analogous subgroups in any locally compact group. These analogous groups in the general setting are called parabolic groups and contraction groups respectively. We will now define them. Let G be a locally compact group and $\alpha \in \mathrm{Aut}(G)$ a bicontinuous automorphism of G. Then, the ***parabolic group*** and ***contraction group*** corresponding to α are respectively defined as:

$$P_\alpha := \{x \in G : \{\alpha^n(x) : n \in \mathbb{N}\} \text{ is relatively compact}\}$$

and

$$U_\alpha := \{x \in G : \alpha^n(x) \to \mathrm{id}_G \text{ as } n \to \infty\}.$$

The ***Levi subgroup*** associated to α is $L_\alpha := P_\alpha \cap P_{\alpha^{-1}}$. In the case when G is a tdlc group, the subgroups P_α and L_α are closed [**151**, Proposition 3], however, U_α is not always closed. It is shown in [**6**] that $P_\alpha = \overline{U_\alpha} \rtimes L_\alpha$. The groups P_α and U_α are important in the theory of tdlc groups, notably due to their involvement in the so called scale theory of tdlc groups [**6**].

In the case that G is a reductive group over a local field, and α is conjugation by an appropriate diagonal matrix, then the groups P_α, U_α and L_α will be respectively parabolic, unipotent and levi subgroups in G in the usual sense. See [**6**, Example 3.13] for further information. This motivates the study of the harmonic analysis of the groups P_α and U_α for general automorphisms α. I initiated this study of harmonic analysis on parabolic and contraction groups in my paper [**27**], and a major part of this thesis is concerned with continuing this investigation.

1.7. Overview of thesis structure and results

This thesis is made up of four chapters, each of which is a paper that has already appeared on the arXiv, and some of which have been published in journals already. In this section we will briefly summarise the main ideas and results of each of the four chapters/papers and connect them to the ideas of this introduction.

1.7.1. Chapter 2: Non-Wiener groups with a Gelfand pair (joint work with Jared White).

As mentioned already in the introduction, it is not well understood which non-discrete tdlc groups are Wiener. The point of this chapter was to make significant progress on this question. In particular, we prove the following theorem in Chapter 2 (see Theorem B).

THEOREM 1.2. *(i) Let X be a connected locally finite graph with infinitely many ends. If $G \leqslant \mathrm{Aut}(X)$ is closed, non-compact and acts transitively on the set of ends of X, then G is not Wiener.*

(ii) Any non-abelian split reductive algebraic group over a non-archimedean local field is not Wiener.

The proof of this result depends on a more general criterion we develop for proving that non-amenable groups with a Gelfand pair are not Wiener (see Theorem A in Chapter 2). This criterion essentially says the following: if G is a non-amenable group with a Gelfand pair, and G has a representation on its Furstenberg boundary which is irreducible and not unitarizable, then G is not Wiener. This chapter thus studies boundary representations of groups with a Gelfand pair carefully and has connections with the recently active literature on this topic [**4, 23, 20, 56**].

The work from this chapter naturally suggests the following conjecture: any non-amenable locally compact group with a Gelfand pair is not Wiener. Some ongoing work outside of this thesis involves seeing how much the assumption of having a Gelfand pair can be removed from this work/conjecture. This conjecture is also further supported by the recent proof that every Hermitian group is amenable [**131**]. As already mentioned in the introduction, the Hermitian and Wiener properties are closely connected.

1.7.2. Chapter 3: Weighted Orlicz $*$-algebras on locally elliptic groups.

As already mentioned, if G is a Hermitian group with polynomial growth, then $L^1(G)$ has a minimal and a maximal ideal for each hull. In particular, a critical part of understanding the ideal theory of $L^1(G)$ for such groups, is understanding the algebras $k(S)/\overline{j(S)}$ for each closed $S \subseteq \widehat{G}$. This has been carried out in quite extensive detail when G is a connected nilpotent real Lie group and S a singleton or empty set.

Now let G be a tdlc contraction group with contractive automorphism α. If U is a compact open subgroup of G, then $G = \bigcup_{i=1}^{\infty} \alpha^{-i}(U)$, so G is an ascending union of compact open subgroups. A locally compact group which is an ascending union of compact open subgroups is called a ***locally elliptic*** group. Locally elliptic groups have polynomial growth but are not compactly generated (or less they are compact). There are also well known examples of locally elliptic groups which are not Hermitian [**53, 71**], and for such groups, it is not known whether the minimal ideal $j(S)$ exists in $L^1(G)$ for general closed $S \subseteq \widehat{G}$.

In this article it is shown that, for any locally elliptic group G, and in particular, any tdlc contraction group, there exists a weight function $\omega : G \to [1, \infty)$ such that $L^1(G, \omega)$ is a Banach $*$-algebra with the weighted L^1-norm, and, $L^1(G, \omega)$ is Hermitian, Wiener, has minimal ideals for each hull, and the representation theory of $L^1(G, \omega)$ is identical to $L^1(G)$. In particular, for locally elliptic groups, a weighted L^1-algebra may be a better choice of group algebra than $L^1(G)$.

The results in this chapter are stated much more generally for (weighted) Orlicz spaces [**127**], which are generalisations of L^p-spaces. If G is a group with polynomial growth, and Φ a Young function, there typically exists a weight ω on G such that the weighted Orlicz space $L^\Phi(G, \omega)$ is a Banach algebra under convolution (see [**80**, Theorem 1.1] and [**110, 111**]. In contrast, $L^p(G)$ is closed under convolution only if G is compact or $p = 1$ [**154**]. One notes that L^p-spaces form a specific class of Orlicz spaces. Indeed, if $\Phi(x) := |x|^p$, then Φ is a Young function and $L^\Phi = L^p$. The following theorem summarises the results of this chapter.

THEOREM 1.3. *Let G be a locally elliptic group, Φ a Young function, and $\omega : G \to [1,\infty)$ a symmetric submultiplicative weight such that $L^{\Phi}(G,\omega)$ is a $*$-subalgebra of $L^1(G)$. The following hold:*

- *(i) $L^{\Phi}(G,\omega)$ is quasi-Hermitian i.e. every self-adjoint continuous compactly supported function in $L^{\Phi}(G,\omega)$ has real spectrum;*
- *(ii) $L^{\Phi}(G,\omega)$ is weakly-Wiener i.e. every proper closed ideal annihilates a simple $L^{\Phi}(G,\omega)$-module;*
- *(iii) If Φ satisfies the Δ_2-condition, then $L^{\Phi}(G,\omega)$ is $*$-regular i.e.* $\mathrm{Prim}_*(L^{\Phi}(G,\omega))$, $\mathrm{Prim}_*(L^1(G))$ *and* $\mathrm{Prim}(C^*(G))$ *are homeomorphic;*
- *(iv) The enveloping C^*-algebra of $L^{\Phi}(G,\omega)$ is $C^*(G)$.*

Furthermore, if $L^{\Phi}(G,\omega)$ is Hermitian, then it is Wiener, and it has a minimal ideal for each hull.

The assumption in part (*iii*) of the above theorem that Φ satisfies the Δ_2-condition is used to guarantee that the continuous compactly supported functions are dense in $L^{\Phi}(G,\omega)$, which is not always the case. This, for example, holds in any L^p-space.

In Chapter 3, it is also shown that, for any locally elliptic group G, and any Young function Φ, there exist many examples of weights ω on G such that $L^{\Phi}(G,\omega)$ is a $*$-subalgebra of $L^1(G)$ and $L^{\Phi}(G,\omega)$ is Hermitian.

1.7.3. Chapter 4: The type I dichotomy for two-step nilpotent locally compact groups (joint work with Pierre-Emmanuel Caprace).

An important result in spectral synthesis is that, if G is a two-step nilpotent real Lie group, then points in $\widehat{G}$ are sets of synthesis. An interesting ongoing question in the harmonic analysis of tdlc groups is to try to extend this result to all two-step nilpotent contraction groups. In particular, if G is a two-step nilpotent contraction group, it is natural to ask whether points in $\widehat{G}$ are sets of synthesis. To do this, one would want to have a better understanding of $\widehat{G}$ for two-step nilpotent contraction groups, and in particular, know whether they are type I.

In Chapter 4, we study the type I problem in greater detail for general two-step nilpotent locally compact groups. We develop a number of algebraic-topological criterions that can be used to determine whether a two-step nilpotent locally compact group is type I. The following theorem states the main criterion that we develop.

THEOREM 1.4. *Let G be a two-step nilpotent second countable locally compact group G with center Z, and π an irreducible unitary representation of G with central character χ. Then the following assertions are equivalent:*

- *(i) π is a closed point in the unitary dual $\widehat{G}$;*

(ii) The homomorphism $\omega_\chi \colon G/Z \to \widehat{G/Z}$*, defined by setting* $\omega_\chi(gZ)(hZ) = \chi([g,h])$*, has a closed image.*

We remark that, if G is a two-step nilpotent locally compact group, then G being type I is equivalent to points in $\widehat{G}$ being closed. Thus, by this criterion, to determine whether a two-step nilpotent group is type I, it suffices to analyse the image of ω_χ for each central character χ.

In Chapter 4, we also apply this result in many settings. Firstly, we tackle the question of whether unipotent groups over positive characteristic local fields are type I and CCR, and provide positive results in the two-step nilpotent case. In particular, we prove the following.

THEOREM 1.5. *Let* k *be a non-discrete locally compact field. The following locally compact two-step nilpotent groups are all type* I*:*

(i) The $2n+1$*-dimensional Heisenberg group over* k*, for all* $n \geqslant 1$*.*
(ii) Given a two-step nilpotent Lie algebra $\mathfrak{g}$ *over* k *with Lie bracket* $[\cdot,\cdot]_{\mathfrak{g}}$*, the unipotent group* $E(\mathfrak{g})$ *with underlying set* $\mathfrak{g}$ *and multiplication defined by* $v.w = v + w + [v,w]_{\mathfrak{g}}$*.*
(iii) The unipotent radical of a minimal k*-parabolic subgroup in an absolutely simple algebraic* k*-group of* k*-rank one.*

As a second application of our criterion, amongst an uncountable class of two-step nilpotent p-torsion contraction groups defined by Glöckner-Willis [**59**, Theorem B], we show that there are uncountably many non-type-I groups and infinitely many type I groups. Thus, this paper demonstrates that the type I problem for tdlc contraction groups is an exceptionally difficult question.

1.7.4. Chapter 5: The affine group of a local field is Hermitian.

At the end of Section 1.3, it was mentioned how the Hermitian property of a group with polynomial growth is related to its spectral synthesis. All compactly generated groups of polynomial growth are Hermitian, however, interestingly, there are non-compactly-generated groups of polynomial growth which are not Hermitian (in particular, the non-Hermitian locally elliptic groups mentioned earlier [**53, 71**]). More generally, research on Hermitian groups has investigated how the growth of a group relates to it having the Hermitian property. Typically groups with exponential growth are not Hermitian groups [**113**], however, there do exist specific examples of Hermitian groups with exponential growth. Indeed, the $ax+b$ group $\mathbb{R} \rtimes \mathbb{R}_{>0}$ is a classical example of a Hermitian group and has exponential growth.

It has been posed as an open question in the literature to determine whether there exist Hermitian tdlc groups with exponential growth (see [**112**, Question 2, page

266] and [**113**, Section 3.6]). A natural example to consider in this direction is the group $\mathbb{Q}_p \rtimes \mathbb{Q}_p^*$. It was unknown whether this group was Hermitian for some time and was stated as an open question in a number of articles (see [**115**, Page 1490] and [**113**, Section 3.6]). In this chapter the following theorem is proved.

THEOREM 1.6. *Let $\mathbb{K}$ be a local field. Then the group $\mathbb{K} \rtimes \mathbb{K}^*$ is Hermitian.*

When $\mathbb{K}$ is a local field, the group $\mathbb{K} \rtimes \mathbb{K}^*$ is not unimodular and hence always has exponential growth. So this paper resolves the question about whether there exist Hermitian tdlc group with exponential growth, and it resolves the question about whether $\mathbb{Q}_p \rtimes \mathbb{Q}_p^*$ is Hermitian.

CHAPTER 2

Non-Wiener groups with a Gelfand pair

with Jared T. White, arXiv:2602.20364.

Abstract

Let G be a non-amenable locally compact group and K a compact subgroup of G such that (G, K) is a Gelfand pair. We show that if G admits a suitable boundary representation which is topologically irreducible and not unitarizable, then G is not a Wiener group in the sense that its Fourier transform does not satisfy the analogue of Wiener's Tauberian theorem. As an application, we show that if G is a closed non-compact boundary transitive group of automorphisms of a connected locally finite graph with infinitely many ends, or a non-abelian split reductive algebraic group over a non-archimedean local field, then G is not Wiener.

2.1. Introduction

In 1932, Norbert Wiener's article entitled *Tauberian Theorems* was published in the Annals of Mathematics [**149**]. This article contains a number of different well known theorems in Fourier analysis, many of which receive the label of "Wiener's Tauberian theorem" in the literature. One of these results can be formulated as follows: a function $f \in L^1(\mathbb{R}^d)$ $(d \geqslant 1)$ generates a dense ideal of $L^1(\mathbb{R}^d)$ if and only if the Fourier transform of f vanishes nowhere. In the terminology of the present article, this means that the group $\mathbb{R}^d$ is a Wiener group.

We shall now discuss how one can formulate a version of this property for general locally compact groups. Let G be a (possibly non-abelian) locally compact group and let $\widehat{G}$ denote the set of unitary equivalences classes of irreducible unitary representations on G. Given $f \in L^1(G)$ and a unitary representation $(\pi, \mathcal{H}_\pi)$ of G, one may define the Fourier transform of f evaluated at π as the operator valued integral

$$\hat{f}(\pi) := \int_G f(x)\pi(x)\,\mathrm{d}x.$$

The integral here is defined in the sense of Bochner and $\hat{f}(\pi)$ is an element of $\mathcal{B}(\mathcal{H}_\pi)$, the space of bounded operators on the Hilbert space $\mathcal{H}_\pi$. Although the Fourier transform of f is defined on all unitary representations of G, we will from here on in consider $\hat{f}$ as a function on the unitary dual $\widehat{G}$ of G, in analogue with the case of abelian groups. Furthermore, when G is an abelian group, the irreducible unitary representations of G can be identified with the unitary characters of G, so the above integral becomes an integral of complex valued functions, and we have the usual Fourier transform on an abelian group.

We remind the reader that there is a bijective correspondence between the (equivalence classes of) irreducible unitary representations of G and the topologically irreducible $*$-representations of the Banach $*$-algebra $L^1(G)$ [**52**, §3.2]. This correspondence is given by the map

$$\widehat{G} \to \widehat{L^1(G)}, \pi \mapsto \widetilde{\pi}$$

where $\widetilde{\pi}(f) := \hat{f}(\pi)$ for each $f \in L^1(G)$.

From the above facts, one can then deduce that the statement that the Fourier transform of $f \in L^1(\mathbb{R}^d)$ vanishes nowhere from the first paragraph is equivalent to requiring that f, and hence the ideal generated by f, is not contained in the kernel of any 1-dimensional $*$-representation of $L^1(G)$. Motivated by these facts, one defines the notion of a Wiener group as follows.

Definition 2.1. A locally compact group G is called ***Wiener*** if every proper closed two-sided ideal $I \subset L^1(G)$ is contained in the kernel of a topologically irreducible $*$-representation of $L^1(G)$.

We remark that the study of Wiener's Tauberian theorem and Wiener groups is intimately related to the topic of spectral synthesis [**10**]. Indeed, given a closed two-sided ideal $I \subseteq L^1(G)$, one defines the following subset of $\widehat{G}$, called the ***hull*** of I:

$$h(I) := \{\pi \in \widehat{G} : \hat{f}(\pi) = 0 \,\forall f \in I\}.$$

Similarly, given $S \subseteq \widehat{G}$ closed, one defines a closed two-sided ideal in $L^1(G)$, called the ***kernel*** of S:

$$k(S) := \{f \in L^1(G) : \hat{f}(\pi) = 0 \,\forall \pi \in S\}.$$

It can be shown that $k(S)$ is the largest closed two-sided ideal $I \subseteq L^1(G)$ satisfying $h(I) = S$. The set $S \subseteq G$ is called a ***set of synthesis*** if $k(S)$ is the only closed ideal in $L^1(G)$ with hull equal to S. The topic of spectral synthesis can be viewed in one way as determining which sets in the unitary dual of a locally compact group are sets of synthesis. The topic of spectral synthesis is an extremely interesting and heavily studied topic, particularly in the cases of abelian groups (see [**65**, Chapter 10], [**130**, Chapter 7] and [**74**, Chapter 6]) and connected nilpotent Lie groups [**98, 99, 94, 9**]. The property of a group G being Wiener is equivalent to the empty set $\varnothing \subset \widehat{G}$ being a set of synthesis.

The question of which groups are Wiener has been investigated extensively, particularly during the mid-to-late 20th century, and the focus during this period was primarily on understanding this property for connected locally compact groups and discrete groups [**88**]. There still remains, however, a strong lack of understanding of which non-discrete totally disconnected locally compact groups (abbreviated ***tdlc groups*** from now on) are Wiener. A main point of this paper is to make further progress on the Wiener property for this class of groups. We note that this article fits into a broader project of the first author concerning progressing the harmonic analysis and representation theory of non-discrete tdlc groups [**27, 28, 29, 22**].

As already mentioned, Wiener's result implies that the group $\mathbb{R}^d$ is Wiener for any $d \geqslant 1$. It is also a well known and classical result in harmonic analysis that every locally compact abelian group is Wiener. We are not sure who this result is originally due to, but one may like to consult [**130**, §7.2] and the references therein for more information. Other classes of groups that are known to be Wiener include nilpotent groups [**95**] and compactly generated groups with polynomial growth [**92**]. In the context of solvable groups, there is exactly one connected exponential (hence solvable) Lie group with real dimension $\leqslant 4$ that is not Wiener [**89**].

On the other hand, as far as the authors are aware, there are no known examples of non-amenable Wiener groups. It is also well known that every non-compact connected semisimple Lie group is not Wiener [**86**, Appendix]. Other than some select examples of solvable Lie groups, these are the only known examples of non-Wiener groups. The point of this article is to develop a general method for showing that a non-amenable group with a Gelfand pair is not Wiener. We use our method to prove that many non-amenable tdlc groups are not Wiener, significantly expanding the class of non-Wiener groups.

The setup in this article is as follows. Let G be a locally compact group and suppose that G contains a compact subgroup K such that the convolution algebra of integrable K-bi-invariant functions on G, denoted by $L^1(K\backslash G/K)$, is commutative. When such a subgroup K exists, the pair (G, K) is called a ***Gelfand pair***. It is shown in [**103**] that there exists a maximal cocompact amenable subgroup P of G such that $G = KP$ and such that we have homeomorphisms

$$\partial G \cong G/P \cong K/K \cap P$$

where ∂G denotes the Furstenberg boundary of G.

Now suppose that G is non-amenable. Then, G/P is non-trivial and compact. The measure on G/P obtained by pushing forward the Haar measure of K via the homeomorphism $G/P \cong K/K \cap P$ is K-invariant and its measure class is preserved by the G-action. Denote this measure by μ. For any complex parameter $z \in \mathbb{C}$, define a representation $\pi_z : G \to \mathcal{B}(L^2(G/P))$ given by

$$\pi_z(g)f(xP) := \left(\frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP)\right)^z f(g^{-1}xP)$$

where $g \in G$ and $f \in L^2(G/P)$. This representation is always strongly continuous.

The main result of this paper is the following theorem. We note that $\mathbb{1}_{G/P}$ denotes the function which is identically 1 on G/P.

THEOREM A. Let G be a non-amenable locally compact group and K a compact subgroup of G such that (G, K) is a Gelfand pair. Suppose that there exists $z \in \mathbb{C}$ with $0 < \operatorname{Re}(z) < 1/2$ such that the representation π_z is topologically irreducible and the matrix coefficient

$$\varphi_z(g) := \langle \pi_z(g)\mathbb{1}_{G/P}, \mathbb{1}_{G/P}\rangle_{L^2(G/P)}$$

is not positive-definite. Then G is not Wiener.

We remark that, by Proposition 2.14, the matrix coefficient φ_z is not positive-definite in the case that the representation π_z is not unitarizable. This fact is very useful in applications of the theorem.

We apply Theorem A to give explicit examples of non-Wiener tdlc groups. To do this, we need to study principal series representations of the following groups deeply, which is completed in the proof of Theorem B.

THEOREM B. (i) Let X be a connected locally finite graph with infinitely many ends. If $G \leqslant \operatorname{Aut}(X)$ is closed, non-compact and acts transitively on the set of ends of X, then G is not Wiener.
(ii) Any non-abelian split reductive algebraic group over a non-archimedean local field is not Wiener.

Part (*i*) of Theorem B resolves an open problem from T. Palmer's well known two volume encyclopaedia on Banach $*$-algebras [**115**] regarding whether the group $\operatorname{Aut}(T)$ is Wiener, where T is a regular tree (see the last line of the table on page 1490).

2.2. Preliminaries on representations and Gelfand pairs

Here we shall layout some basic definitions and results that will be used throughout the article. We assume a rudimentary knowledge of the theory of topological groups, Banach algebras and their various representation theories. We refer the reader to [**145**] or [**137**] for any unproven facts about analysis on Gelfand pairs and representations respectively.

2.2.1. Conventions. Throughout the article, an ideal is understood to be two-sided unless otherwise stated. Integration on a locally compact group G is always performed with respect to some prior fixed left Haar measure.

2.2.2. Gelfand pairs.

We now introduce some of the basic theory of Gelfand pairs which is the centre point of our article.

Let G be a unimodular locally compact group and K a compact subgroup of G. Recall that the space $C_c(G)$ of continuous compactly supported functions on G is a $*$-algebra when equipped with the convolution product and involution $f^*(x) := \overline{f(x^{-1})}$.

Corresponding to the pair (G, K) is the so called ***Hecke algebra*** defined as

$$C_c(K\backslash G/K) := \{f \in C_c(G) : f(kgk') = f(g)\ \forall k, k' \in K, \forall g \in G\}.$$

This is a $*$-subalgebra of $C_c(G)$ consisting of the functions which are ***K-bi-invariant***. Of course, we have natural inclusions $C_c(K\backslash G/K) \subseteq C_c(G) \subseteq L^1(G)$, hence we may also complete the $*$-algebra $C_c(K\backslash G/K)$ in the L^1-norm. This completion, denote by $L^1(K\backslash G/K)$, is a Banach $*$-subalgebra of $L^1(G)$.

Definition 2.2. Let G be a locally compact group and K a compact subgroup of G. The pair (G, K) is called a ***Gelfand pair*** if the algebra $C_c(K\backslash G/K)$ is commutative.

We remark that, if (G, K) is a Gelfand pair, then, since $C_c(K\backslash G/K)$ is dense in $L^1(K\backslash G/K)$, it follows that $L^1(K\backslash G/K)$ is also commutative. Thus $L^1(K\backslash G/K)$ is a commutative Banach $*$-algebra. We shall now say a bit about the dual of $L^1(K\backslash G/K)$. To do this, we must define the notion of a spherical function corresponding to the pair (G, K).

Definition 2.3. Let G be a locally compact group and K a compact subgroup of G such that (G, K) is a Gelfand pair. A ***spherical function*** on the pair (G, K) (or for short, a spherical function on G) is a continuous K-bi-invariant function $\varphi : G \to \mathbb{C}$ such that

$$\chi_\varphi(f) := \int_G f(x)\varphi(x^{-1})\,\mathrm{d}x$$

is a non-trivial multiplicative linear functional of the algebra $C_c(K\backslash G/K)$.

The following result gives a number of equivalent definitions of a spherical function.

Proposition 2.4. [**145**, §6.1(ii)] *Let G be a locally compact group and K a compact subgroup of G such that (G, K) is a Gelfand pair. Let $\varphi : G \to \mathbb{C}$ be a continuous K-bi-invariant function such that $\varphi(\mathrm{id}_G) = 1$. Then the following are equivalent:*

(i) φ is a spherical function;

(ii) For all $x, y \in G$ we have that

$$\int_K \varphi(xky)\,\mathrm{d}k = \varphi(x)\varphi(y)$$

where $\mathrm{d}k$ denotes the normalised Haar measure on K;

(iii) For every $f \in C_c(K\backslash G/K)$*, there exists a complex number* $\lambda(f,\varphi)$ *such that* $f * \varphi = \lambda(f,\varphi)\varphi$*.*

For many examples of Gelfand pairs, the spherical functions can be explicitly determined in a natural way; see for example [**145**, §7].

Now let φ be a spherical function on G. Then, as mentioned above, the function

$$\chi_\varphi(f) = \int_G f(x)\varphi(x^{-1})\,\mathrm{d}x$$

is a multiplicative linear functional on $C_c(K\backslash G/K)$. If φ is furthermore assumed to be bounded, then by density of $C_c(K\backslash G/K)$ in $L^1(K\backslash G/K)$, the functional χ_φ extends to a multiplicative linear functional of $L^1(K\backslash G/K)$ which we will also denote by χ_φ. It is also true that all multiplicative linear functionals of $L^1(K\backslash G/K)$ are formed in this way.

Proposition 2.5. [**145**, Theorem 6.1.7] *Let* G *be a locally compact group and* K *a compact subgroup of* G *such that* (G,K) *is a Gelfand pair. For every bounded spherical function* φ *on* G*, the function*

$$\chi_\varphi(f) = \int_G f(x)\varphi(x^{-1})\,\mathrm{d}x$$

is a multiplicative linear functional on $L^1(K\backslash G/K)$*. Conversely, every multiplicative linear functional on* $L^1(K\backslash G/K)$ *is of this form.*

So we now have a description of the multiplicative linear functionals on $C_c(K\backslash G/K)$ and $L^1(K\backslash G/K)$. We shall now give a description of the $*$-homomorphisms $\chi : L^1(K\backslash G/K) \to \mathbb{C}$. In particular, we need to determine which multiplicative linear functionals χ of $L^1(K\backslash G/K)$ satisfy $\chi(f^*) = \overline{\chi(f)}$ for all $f \in L^1(K\backslash G/K)$. To do this, we need to consider the positive definite spherical functions.

Definition 2.6. Let G be a locally compact group and $\varphi : G \to \mathbb{C}$ a continous function on G. Then φ is called ***positive definite*** if for all $f \in C_c(G)$

$$\int_G \int_G f(x)\overline{f(y)}\varphi(y^{-1}x)\,\mathrm{d}x\,\mathrm{d}y \geqslant 0.$$

We then have the following result.

Proposition 2.7. [**145**, Lemma 5.1.8] *Let* G *be a locally compact group and* K *a compact subgroup of* G *such that* (G,K) *is a Gelfand pair. Let* φ *be a spherical function on* (G,K)*. If* φ *is positive definite, then for all* $g \in G$*,* $\varphi(g) = \overline{\varphi(g^{-1})} =: \varphi^*(g)$*. In particular,* $\chi_\varphi(f^*) = \overline{\chi_\varphi(f)}$ *for all* $f \in C_c(G)$*, so* $\chi_\varphi : L^1(K\backslash G/K) \to \mathbb{C}$ *is a* $*$*-homomorphism if* φ *is positive-definite.*

In a similar light, we also have the following equivalences.

Proposition 2.8. [**153**, Lemma 9.2.5] *Let G be a locally compact group and K a compact subgroup of G such that (G, K) is a Gelfand pair. Let φ be a spherical function on (G, K). Then the following are equivalent:*

(i) *φ is positive definite;*
(ii) *$\chi_\varphi(f * f^*) \geqslant 0$ for all $f \in C_c(K\backslash G/K)$.*

If φ is furthermore assumed to be bounded, then (i) and (ii) are equivalent to:

(iii) *$\chi_\varphi(f * f^*) \geqslant 0$ for all $f \in L^1(K\backslash G/K)$.*

A consequence of Proposition 2.7 is the following.

Corollary 2.9. *Let G be a locally compact group and K a compact subgroup of G such that (G, K) is a Gelfand pair. Suppose that there exists a bounded spherical function φ on G that is not positive definite. Then, the kernel of χ_φ in $L^1(K\backslash G/K)$ is not $*$-closed.*

Proof. Suppose for a contradiction that $\ker_{L^1(K\backslash G/K)}(\chi_\varphi)$ is $*$-closed. Let $f \in \ker_{L^1(K\backslash G/K)}(\chi_\varphi)$. We compute that

$$\begin{aligned}\chi_\varphi(f^*) &= \int_G f^*(x)\varphi(x^{-1})\,\mathrm{d}x = \int_G \overline{f(x^{-1})}\varphi(x^{-1})\,\mathrm{d}x \\ &= \overline{\int_G f(x)\overline{\varphi(x)}\,\mathrm{d}x} = \overline{\int_G f(x)\varphi^*(x^{-1})\,\mathrm{d}x} = \overline{\chi_{\varphi^*}(f)}.\end{aligned}$$

In particular, $f^* \in \ker_{L^1(K\backslash G/K)}(\chi_\varphi)$ if and only if $f \in \ker_{L^1(K\backslash G/K)}(\chi_{\varphi^*})$. Since $\ker_{L^1(K\backslash G/K)}(\chi_\varphi)$ is $*$-closed by assumption, one deduces that $\ker_{L^1(K\backslash G/K)}(\chi_\varphi) = \ker_{L^1(K\backslash G/K)}(\chi_{\varphi^*})$. This implies that the span of χ_φ and the span of χ_{φ^*} are equal in the dual $L^1(K\backslash G/K)^*$. In particular, there exists a constant $\lambda \in \mathbb{C}$ such that $\lambda\chi_\varphi = \chi_{\varphi^*}$. Now let $\mathbb{1}_K \in L^1(K\backslash G/K)$ be the characteristic function on K. Assume that the Haar measure on K has been normalised to have volume 1. Then one computes that $\chi_{\varphi^*}(\mathbb{1}_K) = 1$ and $\lambda\chi_\varphi(\mathbb{1}_K) = \lambda$. Thus $\lambda = 1$ since $\lambda\chi_\varphi = \chi_{\varphi^*}$, and so $\chi_\varphi = \chi_{\varphi^*}$. It follows from this that $\varphi = \varphi^*$. But, by Proposition 2.7, this contradicts the fact that φ is assumed to not be positive definite. $\square$

2.2.3. Smooth and admissible representations.

We start by recalling some definitions. Throughout this section all vector spaces will be over the field of complex numbers. A representation of a locally compact group G is a pair (π, V), where V is a vector space and $\pi : G \to GL(V)$ is a homomorphism.

Definition 2.10. Let G be a tdlc group and (π, V) a representation of G.

(i) A vector $v \in V$ is called ***smooth*** if the subgroup $G_v := \{g \in G : \pi(g)v = v\}$ of G is open.
(ii) The representation π is called ***smooth*** if every $v \in V$ is smooth.

(iii) The representation π is ***admissible*** if for every compact open subgroup $K \leqslant G$, the subspace of fixed vectors $V^K := \{v \in V : \pi(k)v = v \ \forall k \in K\}$ is finite-dimensional.

(iv) Suppose that there is a norm $\|\cdot\|$ on V. Then π is ***continuous*** if for all $v \in V$, the map $g \mapsto \pi(g)v$ is continuous with respect to the group topology on G and the topology induced by the norm on V.

There exists the following equivalent characterisation of a smooth representation. We denote ***the set of compact open subgroups*** of a tdlc group by $\mathrm{COS}(G)$. The proof of the following proposition is obvious after noting that every open subgroup of a tdlc group contains a compact open subgroup.

Proposition 2.11. *Let G be a tdlc group and (π, V) a representation of G. Then the following are equivalent:*

(i) *π is smooth;*
(ii) $V = \bigcup_{K \in \mathrm{COS}(G)} V^K$.

We also note the following well known equivalences of the property of a representation being admissible. See for example, [**137**, §1.5].

Proposition 2.12. *Let G be a tdlc group and (π, V) a smooth representation of G. Then the following are equivalent:*

(i) *(π, V) is admissible;*
(ii) *For every $K \in \mathrm{COS}(G)$, V decomposes as a direct sum of irreducible smooth K-modules each occuring with finite multiplicity.*

Given any representation (π, V) of a tdlc group G, the space $V^\infty := \bigcup_{K \in \mathrm{COS}(G)} V^K$ is a subspace of V invariant under the action of π, and it is precisely the subspace of smooth vectors in V.

Proposition 2.13. *Let G be a tdlc group and (π, V) a smooth admissible representation of G. Let $\|\cdot\|$ be a norm on V with respect to which π is continuous and denote by $(\tilde{\pi}, \tilde{V})$ the completion of this representation with respect to this norm. Then the following are true:*

(i) *$\tilde{V}^K = V^K$ for all compact open subgroups K of V;*
(ii) *$\tilde{V}^\infty = V$;*
(iii) *Every closed G-invariant subspace of $\tilde{V}$ intersects V non-trivially. In particular, (π, V) is algebraically irreducible if and only if $(\tilde{\pi}, \tilde{V})$ is topologically irreducible.*

Proof. *(i)* Since π is a smooth admissible representation, V^K is finite-dimensional for all compact open subgroups $K \in \mathrm{COS}(G)$. Now fix a compact open subgroup

K of G and suppose that $\widetilde{V}^K \neq V^K$. Let $v \in \widetilde{V}^K \backslash V^K$. By density of V in $\widetilde{V}$, there exists a sequence $(v_n)_{n=1}^{\infty} \subset V$ such that $v_n \to v$ as $n \to \infty$. Then, since the projection

$$P^K : \widetilde{V} \to \widetilde{V}^K, v \mapsto \int_K \widetilde{\pi}(k) v \, \mathrm{d}k$$

is continuous, it follows that $P^K(v_n) \to P^K(v) = v$ as $n \to \infty$. But since $v_n \in V$ for each n, $P^K(v_n) \in V^K$ for each n. So we have found a sequence in V^K converging to v in $\widetilde{V}$. But V^K being finite-dimensional must be closed in $\widetilde{V}$, so $v \in V^K$.

(ii) By part *(i)*, $\widetilde{V}^K = V^K$ for each $K \in \mathrm{COS}(G)$. Thus $\widetilde{V}^{\infty} = \bigcup_{K \in \mathrm{COS}(G)} \widetilde{V}^K = \bigcup_{K \in \mathrm{COS}(G)} V^K = V$.

(iii) Suppose that $W \subseteq \widetilde{V}$ is a closed G-invariant subspace and let $w \in W$ be non-zero. To show that W intersects V non-trivially, it suffices to show that there exists a compact open subgroup $K \in \mathrm{COS}(G)$ such that $P^K(w) \neq 0$. Suppose for a contradiction that this is not the case, that is, for every $K \in \mathrm{COS}(G)$, $P^K(w) = 0$.

Let μ denote the Haar measure on G. Let $X := \{K \in \mathrm{COS}(G) : \mu(K) \leqslant 1\}$. Note that X is a directed set when equipped with reverse set inclusion as the ordering. Then, $(P^K(w))_{K \in X}$ is a net in $\widetilde{V}$. We claim that it converges to w. Indeed, for any $K \in X$

$$\begin{aligned} \left\| P^K(w) - w \right\| &= \left\| \int_K \widetilde{\pi}(k) w \, \mathrm{d}k - w \right\| \\ &\leqslant \sup_{k \in U} \left\| \widetilde{\pi}(k) w - w \right\| . \end{aligned}$$

So by continuity of $\widetilde{\pi}$ (at the identity of G), for every $\epsilon > 0$, there exists a compact open subgroup $K_0 \in X$ such that

$$\left\| P^{K_0}(w) - w \right\| \leqslant \sup_{k \in K_0} \left\| \widetilde{\pi}(k) w - w \right\| < \epsilon.$$

This implies that the net $(P^K(w))_{K \in X}$ converges to w. Since w is assumed to be non-zero, there must exist a compact open subgroup $K \in X$ such that $P^K(w) \neq 0$. This implies that $P^K(\widetilde{V}) \subseteq V^K$ intersects W non-trivially.

To prove the second claim of *(iii)*, suppose that (π, V) is algebraically irreducible, and let W be a closed invariant subspace of $\widetilde{V}$. Then by the previous argument, $V \cap W$ is a non-trivial invariant subspace of V, so we must have that $V \subseteq W$ since V is algebraically irreducible. This implies that $W = \widetilde{V}$ since W is closed and V is dense in $\widetilde{V}$. So $(\widetilde{\pi}, \widetilde{V})$ is topologically irreducible.

To prove the converse, suppose that $(\widetilde{\pi}, \widetilde{V})$ is topologically irreducible. Then, let W be a non-trivial G-invariant subspace of V. Note that W is smooth and admissible since it is a subrepresentation of a representation satisfying these properties. We need to show that $W = V$. Since the closure of W in $\widetilde{V}$ is a closed non-trivial G-invariant subspace of $\widetilde{V}$, it follows that W must be dense in $\widetilde{V}$, since $\widetilde{V}$ is topologically irreducible. Then, for every compact open subgroup K of G, the projection P^K is continuous. Thus it follows that $P^K(W) = W^K$ is dense in $P^K(\widetilde{V}) = \widetilde{V}^K = V^K$ for every $K \in \mathrm{COS}(G)$. But W^K is finite-dimensional since W is admissible, so W^K must be closed, hence $W^K = V^K$ for every $K \in \mathrm{COS}(G)$. But since both W and V are smooth, we have that $W = \bigcup_{K \in \mathrm{COS}(G)} W^K = \bigcup_{K \in \mathrm{COS}(G)} V^K = V$. This implies that V is algebraically irreducible and completes the proof. $\square$

Finally, we have the following result which will be used later in the article.

Proposition 2.14. *Let G be a tdlc group and (π, V) a smooth admissible algebraically irreducible representation of G. Let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ be an inner-product on V with respect to which π is continuous but not necessarily unitary. Suppose that there exists a $v \in V$ such that the matrix coefficient $\varphi(g) := \langle \pi(g)v, v \rangle$ is positive-definite. Then, the representation (π, V) must be unitarizable.*

Proof. Since $\mathrm{span}\{\pi(g)v : g \in G\}$ is a G-invariant subspace of V, and since (π, V) is algebraically irreducible, we must have that $V = \mathrm{span}\{\pi(g)v : g \in G\}$. In particular, every element of V can be written as a sum of the form $\sum_{i=1}^n c_i \pi(g_i) v$ with $c_i \in \mathbb{C}$ and $g_i \in G$ for each i.

Now define a bilinear form $B : V \times V \to \mathbb{C}$ by

$$B\left(\sum_{i=1}^n c_i \pi(g_i) v, \sum_{j=1}^m d_j \pi(h_j) v \right) := \sum_{i,j} c_i \overline{d_j} \varphi(h_j^{-1} g_i).$$

One checks easily that B is G-invariant. It follows from Proposition 2.8 that B is Hermitian and B is positive since φ is positive-definite. We now claim that B is definite. Indeed, $\ker(B) := \{w \in V : B(w, w) = 0\}$ is a proper G-invariant subspace of V by [**52**, Theorem A.3], so $\ker(B)$ must be trivial since (π, V) is algebraically irreducible. Thus B is a G-invariant positive-definite Hermitian form on V, in particular a G-invariant inner-product, so (π, V) is unitarizable. $\square$

2.2.4. Contragradient representations and matrix coefficients.

Here we introduce some notation concerning contragradient representations and matrix coefficients.

Let G be a tdlc group and (π, V) a smooth representation of G. Throughout this section and the remainder of the article, $\widehat{V}$ will be used to denote the (complex) vector space dual of V. There is a representation $\widehat{\pi} : G \to GL(\widehat{V})$ given by

$$\widehat{\pi}(g)\widehat{v}(v) = \widehat{v}(\pi(g^{-1})v)$$

for $\widehat{v} \in \widehat{V}$, $v \in V$ and $g \in G$.

Corresponding to the representation (π, V) is also the complex conjugate representation denoted by $(\overline{\pi}, \overline{V})$. Here $\overline{V}$ is the vector space V where we have defined a new scalar multiplication by $c \cdot_{\text{new}} v := \overline{c} \cdot_{\text{old}} v$ for $c \in \mathbb{C}$ and $v \in V = \overline{V}$. The representation $\overline{\pi}$ then acts on $\overline{V}$ by $\overline{\pi}(g)v = \pi(g)v$ for $v \in \overline{V}$.

Definition 2.15. Let G be a tdlc group and (π, V) a smooth representation of G. We use the following terminology:

(i) The ***contragradient*** representation of (π, V), denoted by $(\widetilde{\pi}, \widetilde{V})$, is the subrepresentation of smooth vectors in $(\widehat{\pi}, \widehat{V})$;
(ii) The ***Hermitian contragradient*** representation of (π, V), denoted by (π^+, V^+), is the complex conjugate representation of $(\widetilde{\pi}, \widetilde{V})$.

The concept of contragradient representations allows us to formulate a type of matrix coefficient for general smooth representations.

Definition 2.16. Let G be a tdlc group and (π, V) a smooth representation of G. Given $\widetilde{v} \in \widetilde{V}$ and $v \in V$, a function of the form

$$\langle \pi(g)v, \widetilde{v} \rangle := \widetilde{v}(\pi(g)v)$$

will be called a ***matrix coefficient*** of the representation (π, V).

2.2.5. Induced representations. Let G be a tdlc group and H a closed subgroup of G. In this subsection we discuss how to induce a smooth representation (σ, W) of H to a smooth representation of G. There are multiple types of induction that will be discussed. We refer the reader to [**137**, §1.7] for further information.

Fix a smooth representation (σ, W) of H for the remainder of this section. Define a space of functions associated to this representation by

$$\begin{aligned} V^\sigma := \{f : G \to W : f(gh) =& \sigma(h^{-1})f(g) \; \forall h \in H, \\ & \exists U \in \mathrm{COS}(G) \text{ s.t. } f(ug) = f(g) \; \forall u \in U \; \forall g \in G\}. \end{aligned}$$

We may also consider the subspace of compactly supported functions in V^σ which we will denote by V^σ_c. Then, the ***induced representation*** (resp. ***compactly induced representation***) of σ to G is denoted by $\mathrm{ind}_H^G(\sigma)$ (resp. c-$\mathrm{ind}_H^G(\sigma)$) and acts on V^σ (resp. V^σ_c) by the left regular representation of G.

One notes that the representations $\mathrm{ind}_H^G(\sigma)$ and c-$\mathrm{ind}_H^G(\sigma)$ are isomorphic if H is cocompact in G i.e. the coset space G/H is compact.

2.2.6. Boundary representations and the Iwasawa decomposition. In this subsection we recall some definitions and facts about boundary representations on groups with a Gelfand pair. We also note down some results and discussion from the following paper of Nicolas Monod [**103**].

We summarise in the following statement the main results of the paper [**103**].

Theorem 2.17. [**103**] *Let G be a locally compact group and $K \leqslant G$ a compact subgroup such that (G, K) is a Gelfand pair. Then, there exists a cocompact amenable subgroup $P \leqslant G$ such that G admits an Iwasawa decomposition $G = KP$. Furthermore, if P is chosen to be maximal, then we have homeomorphisms $\partial G \cong G/P \cong K/K \cap P$, where ∂G denotes the Furstenberg boundary of G.*

Suppose that we have a Gelfand pair (G, K) where G is non-amenable and choose P a maximal cocompact amenable subgroup of G such that $G = KP$. Let μ denote the measure on G/P obtained by pushing forward the normalised Haar measure on K via the homeomorphism $G/P \cong K/K \cap P$. The measure μ on G/P is K-invariant and the G-action preserves the measure class of μ. However, μ is not G-invariant.

Now let Δ_P denote the modular function on P. This modular function is always non-trivial as a consequence of assuming that G is non-amenable (see, for example, [**52**, Theorem 2.51] for more details). Define a function ρ on G by $\rho(kp) := \Delta_P(p)$ for $k \in K$ and $p \in P$. Then, by the results of [**52**, §2.6], ρ is a continuous function on G and we have that

$$\frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP) = \frac{\rho(g^{-1}x)}{\rho(x)}.$$

One should note that the function defined by

$$c : G \times G/P \to \mathbb{R}_{>0}, (g, xP) \mapsto \frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP) = \frac{\rho(g^{-1}x)}{\rho(x)}$$

is well known to satisfy the cocycle identity.

We now define the following boundary representations, maintaining the notation as given above. In the following definition, given $p \in [1, \infty)$, we denote by $\mathcal{B}(L^p(G/P))$ the space of bounded operators on the Banach space $L^p(G/P)$.

Definition 2.18. Given $p \in [1, \infty)$ and $z \in \mathbb{C}$, we define a representation $\pi_{z,p} : G \to \mathcal{B}(L^p(G/P))$ whose action on $f \in L^p(G/P)$ is given by

$$\pi_{z,p}(g)f(xP) := \left(\frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP)\right)^z f(g^{-1}xP)$$

for $g, x \in G$. In the case when $p = 2$, we will denote the representation by π_z instead of $\pi_{z,2}$.

From now on, any representation of the form of a representation in the above definition will be referred to as a ***boundary representation*** on the Gelfand pair (G, K). We remark that for a general complex number z, a boundary representation need not be unitary nor irreducible, even in the case when $p = 2$.

We now show that these boundary representations are continuous.

Proposition 2.19. *For every $p \in [1, \infty)$ and $z \in \mathbb{C}$, the representation $(\pi_{z,p}, L^p(G/P))$ is (strongly) continuous i.e. for all $f \in L^p(G/P)$, the map*

$$G \to L^p(G/P), g \mapsto \pi_{z,p}(g)f$$

is continuous with respect to the group topology on G and the norm topology on $L^p(G/P)$.

Proof. Let U be a symmetric compact neighbourhood of the identity in G. We first claim that the map $g \mapsto \|\pi_{z,p}(g)\|_{\mathcal{B}(L^p(G/P))}$ is bounded on U. Indeed, since the function ρ is continuous, and $U \times G/P$ is a compact subset of $G \times G/P$, the map $(g, xP) \mapsto \frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP) = \frac{\rho(g^{-1}x)}{\rho(x)}$ is bounded by some constant $M > 0$ on $U \times G/P$. Then, we have that, for any $f \in L^p(G/P)$ and $g \in U$

$$\begin{aligned} \|\pi_{z,p}(g)f\|_{L^p(G/P)} &= \left(\int_{G/P} \left| \frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP) f(g^{-1}xP) \right|^p \mathrm{d}\mu(xP) \right)^{1/p} \\ &\leqslant M \left(\int_{G/P} \left| f(g^{-1}xP) \right|^p \mathrm{d}\mu(xP) \right)^{1/p} \\ &= M \left(\int_{G/P} \left| f(xP) \right|^p \frac{\mathrm{d}g^{-1}\mu}{\mathrm{d}\mu}(xP) \, \mathrm{d}\mu(xP) \right)^{1/p} \\ &\leqslant M^{\frac{p+1}{p}} \left(\int_{G/P} \left| f(xP) \right|^p \mathrm{d}\mu(xP) \right)^{1/p} \\ &= M^{\frac{p+1}{p}} \|f\|_{L^p(G/P)} . \end{aligned}$$

Thus, by definition of the operator norm, we have that $\|\pi_{z,p}(g)\|_{\mathcal{B}(L^p(G/P))} \leqslant M^{\frac{p+1}{p}}$ for all $g \in U$.

Now, to prove the proposition, it suffices to show that $\pi_{z,p}$ is continuous at the identity. In particular, we need to show that, for every $f \in L^p(G/P)$, and every $\epsilon > 0$, there exists a neighbourhood W of the identity in G such that

$$\|\pi_{z,p}(g)f - f\|_{L^p(G/P)} \leqslant \epsilon$$

for all $g \in W$. So fix $f \in L^p(G/P)$ and $\epsilon > 0$ for the remainder of the proof. We will now find such a neighbourhood W.

Since G/P is compact, the continuous functions on G/P are dense in $L^p(G/P)$. So we may find a continuous function f' on G/P such that

$$\|f' - f\|_{L^p(G/P)} \leqslant \frac{\epsilon}{2(M^{\frac{p+1}{p}} + 1)}.$$

Also, by continuity of ρ and uniform continuity of f', we may find neighbourhoods U_1 and U_2 of the identity in G such that

$$\left|\left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^{z} f'(g^{-1}xP) - f'(g^{-1}xP)\right| \leqslant \frac{\epsilon}{4}.$$

for all $(g, xP) \in U_1 \times G/P$ and

$$|f'(g^{-1}xP) - f'(xP)| \leqslant \frac{\epsilon}{4}$$

for all $(g, xP) \in U_2 \times G/P$. Now let λ_G denote the left regular representation of G. Then, for all $g \in U_1 \cap U_2$, we have that

$$\begin{aligned}
\left\|\pi_{z,p}(g)f' - f'\right\|_{L^p(G/P)} &= \left\|\pi_{z,p}(g)f' - \lambda_G(g)f' + \lambda_G(g)f' - f'\right\|_{L^p(G/P)} \\
&\leqslant \left\|\pi_{z,p}(g)f' - \lambda_G(g)f'\right\|_{L^p(G/P)} + \left\|\lambda_G(g)f' - f'\right\|_{L^p(G/P)} \\
&= \left(\int_G \left|\left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^{z} f'(g^{-1}xP) - f'(g^{-1}xP)\right|^p \mathrm{d}\mu(xP)\right)^{1/p} \\
&\qquad + \left(\int_G \left|f'(g^{-1}xP) - f'(xP)\right|^p \mathrm{d}\mu(xP)\right)^{1/p} \\
&\leqslant \left(\frac{\epsilon^p}{4^p}\right)^{1/p} + \left(\frac{\epsilon^p}{4^p}\right)^{1/p} = \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

Finally, setting $W := U \cap U_1 \cap U_2$, we have that, for all $g \in W$,

$$\begin{aligned}
&\|\pi_{z,p}(g)f - f\|_{L^p(G/P)} \\
&\leqslant \left\|\pi_{z,p}(g)f - \pi_{z,p}(g)f'\right\|_{L^p(G/P)} + \left\|\pi_{z,p}(g)f' - f'\right\|_{L^p(G/P)} + \left\|f' - f\right\|_{L^p(G/P)} \\
&\leqslant (\|\pi_{z,p}(g)\|_{\mathcal{B}(L^p(G/P))} + 1) \left\|f - f'\right\|_{L^p(G/P)} + \left\|\pi_{z,p}(g)f' - f'\right\|_{L^p(G/P)} \\
&\leqslant (M^{\frac{p+1}{p}} + 1)\frac{\epsilon}{2(M^{\frac{p+1}{p}} + 1)} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

This completes the proof. □

The point of the remainder of this subsection is to show that if G is a tdlc group then the subrepresentation of smooth vectors in a boundary representation is naturally isomorphic to parabolically inducing a certain character from P. So for the remainder of this subsection, we assume that G is tdlc group.

Now let $z \in \mathbb{C}$ and consider the representation $\sigma_z := \mathrm{ind}_P^G(\Delta_P^{-z})$, where the induction is as defined in the last section. Since P is cocompact in G by definition, we also have that $\sigma_z \cong \text{c-ind}_P^G(\Delta_P^{-z})$.

Let V_z denote the representation space of the representation σ_z. By definition of V_z in the last section, we may restrict functions in V_z to K. One checks that restricting functions in V_z to K defines a map

$$R_K : V_z \to C^\infty(K)$$

where $C^\infty(K)$ denotes the space of locally constant functions on K. The image of this map is precisely those functions in $C^\infty(K)$ who are right translation invariant by $K \cap P$. Thus, R_K may in fact be seen as a map of the form

$$R_K : V_z \to C^\infty(K/K \cap P)$$

where we view $C^\infty(K/K \cap P)$ as a subspace of $C^\infty(K)$. Since our group G admits an Iwasawa decomposition $G = KP$, one can check using this that R_K is a bijection and hence identifies V_z with $C^\infty(K/K \cap P)$. Consequently, the isomorphism type of the representation space V_z does not depend on the complex parameter z.

Define a norm on V_z by

$$\|f\|_{K,p} := \left(\int_K |R_K(f)(k)|^p \, \mathrm{d}k \right)^{1/p}$$

for $f \in V_z$. The completion of V_z with respect to this norm is a Banach space which we denote by $E_{z,p}$. The map R_K extends by continuity to $E_{z,p}$ and gives an isomorphism of $E_{z,p}$ with $L^p(K/K \cap P)$ i.e. a linear isometric isomorphism

$$R_K : E_{z,p} \to L^p(K/K \cap P).$$

Also, since σ_z acts on V_z via the left regular representation, it is continuous with respect to the norm $\| \cdot \|_{K,p}$. Thus we may extend the representation σ_z to $E_{z,p}$ by continuity, and the resulting representation of G will be denoted by $\sigma_{z,p}$.

We define the following map which we shall use in the following proposition:

$$\varphi : K/K \cap P \to G/P, k(K \cap P) \mapsto kP.$$

The map φ is a homeomorphism. Then, maintaining the notation as laid out in this subsection, we have the following.

Proposition 2.20. *For every $p \in [1, \infty)$ and $z \in \mathbb{C}$, the linear map*

$$U_{z,p} : E_{z,p} \to L^p(G/P), f \mapsto R_K(f) \circ \varphi^{-1}$$

is an isometric isomorphism of Banach spaces. Furthermore, it intertwines the representations $\pi_{z,p}$ and $\sigma_{z,p}$ i.e.

$$U_{z,p}\sigma_{z,p}(g) = \pi_{z,p}(g)U_{z,p}$$

for all $g \in G$. Thus $\sigma_{z,p} \cong \pi_{z,p}$.

Proof. One checks easily from the definitions that $U_{z,p}$ is an isometric isomorphism. We now show that $U_{z,p}$ intertwines the representations $\pi_{z,p}$ and $\sigma_{z,p}$. It suffices to show, by density of V_z in $E_{z,p}$, that given $f \in V_z$,

$$U_{z,p}(\sigma_{z,p}(g)(f)) = \pi_{z,p}(g)(U_{z,p}(f))$$

for all $g \in G$. Let $x \in G$ and write $x = kp$ for some $k \in K$ and $p \in P$. Note that $U_{z,p}(f)(xP) = f(k)$. Given $g \in G$, write $g^{-1}k = ab$ for $a \in K$ and $b \in P$. Then, $g^{-1}x = g^{-1}kp = abp$. Since $bp \in P$, we have that

$$\rho(g^{-1}x) = \Delta_P(bp) = \Delta_P(b)\Delta_P(p) = \rho(g^{-1}k)\rho(x)$$

from which it follows that $\rho(g^{-1}k) = \frac{\rho(g^{-1}x)}{\rho(x)}$. We then compute that

$$\begin{aligned} U_{z,p}(\sigma_{z,p}(g)(f))(xP) &= (\sigma_{z,p}(g)(f))(k) = f(g^{-1}k) = f(ab) \\ &= \Delta_P^{-z}(b^{-1})f(a) = \Delta_P^{z}(b)f(a) = \rho(g^{-1}k)^z f(a) = \left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^z f(a). \end{aligned}$$

Similarly, we compute that

$$\begin{aligned} (\pi_z(g)(U_{z,p}(f)))(xP) &= \left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^z U_{z,p}(f)(g^{-1}xP) \\ &= \left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^z U_{z,p}(f)(g^{-1}kP) = \left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^z f(a). \end{aligned}$$

Thus it follows that $U_{z,p}\sigma_{z,p}(g) = \pi_{z,p}(g)U_{z,p}$ on the dense subspace V_z for all $g \in G$. By continuity, it follows that $U_{z,p}$ intertwines $\sigma_{z,p}$ and $\pi_{z,p}$. This completes the proof. □

2.2.7. A theorem of Godement.

In this section we discuss a result from [**61**] that we will use in the proof of Theorem A. Some of the notation and terminology in [**61**] is dated and requires some effort to understand, so we do this to clarify the result from Godement we use and to clarify our proof of Theorem A.

Let G be a locally compact group and $K \leqslant G$ a compact subgroup. Given $\sigma \in \widehat{K}$, we write $p_\sigma := \dim(\sigma)\,\overline{\chi_\sigma}$, where $\chi_\sigma := \mathrm{Tr}(\sigma)$ is the character of σ. Note that $p_\sigma \in L^1(K)$ is a self-adjoint idempotent.

Let π be a topologically irreducible representation of G on a Banach space E and define

$$P_\sigma := \pi(p_\sigma) = \int_K \pi(k)p_\sigma(k)\,\mathrm{d}k.$$

The operator P_σ is a projection of E onto a closed subspace that we denote by $E(\sigma)$. The subspace $E(\sigma)$ is precisely the σ-isotypic subspace of the representation $\pi|_K$.

Godement proves the following fact in his article (see [**61**, §8]).

Proposition 2.21. [**61**, §8] *Let G be a locally compact group and K a compact subgroup of G. Let π be a topologically irreducible uniformly bounded representation of G on a Banach space E. Suppose that for every $\sigma \in \widehat{K}$ we have $\dim(E(\sigma)) < \infty$. Then $\ker_{L^1(G)}(\pi)$ is maximal amongst the collection of closed ideals of $L^1(G)$.*

Remark 2.22. In a unital Banach algebra, maximal ideals are automatically closed, so maximal and maximal closed are the same thing. However, in the non-unital setting, it can happen that an ideal is maximal closed but not maximal in the set of all proper ideals (see [**61**, pg. 515-516].)

We now show that if (G, K) is a Gelfand pair and $\pi_{z,p}$ is a boundary representation of G for some $z \in \mathbb{C}$, then the finite-dimensionality assumption of Proposition 2.21 is satisfied.

To do this, let G be a locally compact group and $p \in [1, \infty)$. The left-regular representation of G on $L^p(G)$ will be denoted by $\lambda_{G,p}$.

Lemma 2.23. *Let K be a compact group, $\sigma \in \widehat{K}$ and $p \in [1, \infty)$. Then $\lambda_{K,p}(p_\sigma)$ is finite rank.*

Proof. Given $\tau \in \widehat{K}$, let F_τ be the span of the matrix coefficients of τ inside $L^p(K)$. Since τ must be finite dimensional, the subspace F_τ is finite dimensional, hence F_τ is closed. By [**52**, Theorem 5.11], span $\bigcup_{\tau\in\widehat{K}} F_\tau$ is dense in $L^p(K)$. Now consider a finite linear combination $f := \sum_{i=1}^n f_{\tau_i}$ where $\tau_1, \ldots, \tau_n \in \widehat{K}$ and $f_{\tau_i} \in F_{\tau_i}$ for each i. Then the orthogonality relations imply that

$$\lambda_{K,p}(p_\sigma)f = d_\sigma \sum_{i=1}^n \overline{\chi_\sigma} * f_{\tau_i} = d_\sigma \sum_{i=1}^n \chi_{\overline{\sigma}} * f_{\tau_i} \in F_{\overline{\sigma}}.$$

Now, a generic $f \in L^p(K)$ can be written as a limit $f = \lim_\beta f_\beta$, where each $f_\beta \in$ span $\bigcup_{\sigma\in\widehat{K}} F_\sigma$. Then for each index β we have $\lambda_{K,p}(p_\sigma)f_\beta \in F_{\overline{\sigma}}$ by the above, so that $\lambda_{K,p}(p_\sigma)f = \lim_\beta \lambda_{K,p}(p_\sigma)f_\beta \in F_{\overline{\sigma}}$ as well. Hence the image of $\lambda_{K,p}(p_\sigma)$ is contained in $F_{\overline{\sigma}}$, which proves the result. ☐

Proposition 2.24. *Let G be a locally compact group and K a compact subgroup of G such that (G, K) is a Gelfand pair. Let P be a maximal cocompact amenable subgroup of G such that $G = KP$. For $z \in \mathbb{C}$ and $p \in [2, \infty)$, consider the boundary representation $\pi_{z,p}$ of G on $E := L^p(G/P)$. Then, for every $\sigma \in \widehat{K}$, $\dim(E(\sigma)) < \infty$.*

Proof. Recall from Section 2.2.6 that we have a homeomorphism $G/P \cong K/K \cap P$. Also, note that, for any $k \in K$ and $x \in G$, $\rho(k^{-1}x) = \rho(x)$. Thus, for $k \in K$ and $f \in L^p(G/P)$ we have

$$\pi_{z,p}(k)f(xP) = f(k^{-1}xP) \tag{2.2.1}$$

since $\frac{\rho(k^{-1}x)}{\rho(x)} = 1$. Also, since we have a homeomorphism $G/P \cong K/K \cap P$, $L^p(G/P)$ may be identified with $L^p(K/K \cap P)$, which in turn may be identified with the subspace of $L^p(K)$ consisting of functions that are constant on cosets of $K \cap P$. Under this identification, equation (2.2.1) tells us that the action of $\pi_{z,p}$

when restricted to K coincides with the action of the left-regular representation $\lambda_{K,p}$ on the subspace $L^p(K/K \cap P)$. Thus, given $\sigma \in \widehat{K}$, we have $\operatorname{rank}(\pi_{z,p}(p_\sigma)) \leqslant \operatorname{rank}(\lambda_{K,p}(p_\sigma)) < \infty$. The proposition then follows immediately. □

2.3. Proof of Theorem A

Throughout this section we assume the hypotheses and notation specified in Theorem A.

We start with the following lemma. A Banach $*$-algebra $\mathcal{A}$ is called ***Wiener*** if every proper closed ideal of $\mathcal{A}$ is contained in the kernel of a topologically irreducible non-degenerate $*$-representation of $\mathcal{A}$ [**115**, Definition 11.5.3]. This just generalises the definition given for $L^1(G)$ in the introduction.

Lemma 2.25. *Suppose that $\mathcal{A}$ is a Wiener Banach $*$-algebra. Then every maximal closed ideal of $\mathcal{A}$ is $*$-closed.*

Proof. Suppose that $\mathcal{A}$ has the Wiener property and let I be a maximal closed ideal of $\mathcal{A}$. According to the Wiener property, there exists a non-trivial $*$-representation π of $\mathcal{A}$ with $I \subseteq \ker(\pi)$. By maximality of I, we must have that $\ker(\pi) = I$. Since π is a $*$-representation, $\ker(\pi)$ is $*$-closed, and so I is $*$-closed. □

We now prove some lemmas about boundary representations and their matrix coefficients.

Lemma 2.26. *Fix $z \in \mathbb{C}$ with $0 < \operatorname{Re}(z) < 1$ and define $p := 1/\operatorname{Re}(z)$. The representation $(\pi_{z,p}, L^p(G/P))$ is an isometric representation.*

Proof. Indeed, for any $f \in L^p(G/P)$, one makes the following computation:

$$\begin{aligned}
\|\pi_{z,p}(g)f\|^p_{L^p(G/P)} &= \int_{G/P} \left|\pi_{z,p}(g)f(xP)\right|^p \mathrm{d}\mu(xP) \\
&= \int_{G/P} \left|\left(\frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP)\right)^z f(g^{-1}xP)\right|^p \mathrm{d}\mu(xP) \\
&= \int_{G/P} \left(\frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP)\right)^{p\operatorname{Re}(z)} |f(g^{-1}xP)|^p \,\mathrm{d}\mu(xP) \\
&= \int_{G/P} \frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP)|f(g^{-1}xP)|^p \frac{\mathrm{d}\mu}{\mathrm{d}g\mu}(xP)\,\mathrm{d}\mu(g^{-1}xP) \\
&= \int_{G/P} |f(g^{-1}xP)|^p \,\mathrm{d}\mu(g^{-1}xP) = \|f\|^p_{L^p(G/P)}\,.
\end{aligned}$$

□

Lemma 2.27. *For any $z \in \mathbb{C}$ with $0 < \operatorname{Re}(z) < 1$, the matrix coefficient φ_z is bounded.*

Proof. Set $p := 1/\mathrm{Re}(z)$. By the previous lemma, the representation $(\pi_{z,p}, L^p(G/P))$ is an isometric representation. Now consider $\mathbb{1}_{G/P}$ simultaneously as the characteristic function on G/P (which is an element of $L^p(G/P)$) and as the constant 1 linear functional in $L^p(G/P)^* \cong L^q(G/P)$, where q is the Hölder conjugate of p. Then, the corresponding matrix coefficient as defined in Definition 2.16 is given by

$$\langle \pi_{z,p}(g)\mathbb{1}_{G/P}, \mathbb{1}_{G/P}\rangle = \int_{G/P} \left(\frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(xP)\right)^z \mathrm{d}\mu(xP).$$

But one checks easily that this matrix coefficient is precisely equal to the matrix coefficient $\varphi_z(g) = \langle \pi_{z,2}(g)\mathbb{1}_{G/P}, \mathbb{1}_{G/P}\rangle_{L^2(G/P)}$. The result then follows from the fact that matrix coefficients of an isometric Banach space representation are bounded. □

Lemma 2.28. *Let $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$. Set $p := 1/\mathrm{Re}(z)$. If the representation $(\pi_{z,2}, L^2(G/P))$ is topologically irreducible, then the representation $(\pi_{z,p}, L^p(G/P))$ is also topologically irreducible.*

Proof. Assume that $(\pi_{z,2}, L^2(G/P))$ is topologically irreducible. Then, since $p > 2$ and G/P is compact, $L^p(G/P) \subseteq L^2(G/P)$. This implies that $L^p(G/P)^\infty \subseteq L^2(G/P)^\infty$. By Proposition 2.13(iii), since $L^2(G/P)$ is topologically irreducible, $L^2(G/P)^\infty$ is algebraically irreducible. But $L^p(G/P)^\infty$ is a non-trivial submodule of $L^2(G/P)^\infty$, hence, $L^p(G/P)^\infty = L^2(G/P)^\infty$. Thus $L^p(G/P)^\infty$ is algebraically irreducible, and by applying Proposition 2.13(iii) again, we get that $(\pi_{z,p}, L^p(G/P))$ is topologically irreducible. □

We now prove Theorem A.

Proof of Theorem A. Suppose the hypotheses of Theorem A. Fix $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$ such that the representation $(\pi_{z,2}, L^2(G/P))$ is topologically irreducible and the matrix coefficient $\varphi_z(g) := \langle \pi_{z,2}(g)\mathbb{1}_{G/P}, \mathbb{1}_{G/P}\rangle_{L^2(G/P)}$ is not positive definite. Set $p := 1/\mathrm{Re}(z) \in (2, \infty)$. Then, by Lemma 2.26 and Lemma 2.28, $(\pi_{z,p}, L^p(G/P))$ is an isometric representation and it is topologically irreducible.

Since $(\pi_{z,p}, L^p(G/P))$ is an isometric representation, it extends to a representation of $L^1(G)$ which we also denote by $(\pi_{z,p}, L^p(G/P))$. Furthermore, by Proposition 2.24, it follows that the representation $(\pi_{z,p}, L^p(G/P))$ satisfies the hypotheses of Proposition 2.21 and thus the kernel of $(\pi_{z,p}, L^p(G/P))$ as a representation of $L^1(G)$, denoted by $I := \ker_{L^1(G)}(\pi_{z,p})$, is a maximal closed ideal in $L^1(G)$. To complete the proof of Theorem A, by Lemma 2.25, it suffices to show that the ideal I is not $*$-closed.

To do this, note that by Lemma 2.27, the matrix coefficient φ_z is bounded and hence defines a character χ_{φ_z} of $L^1(K\backslash G/K)$ whose kernel, denoted $J := \ker_{L^1(K\backslash G/K)}(\chi_{\varphi_z})$, is not $*$-closed by Corollary 2.9. To complete the proof, it suffices to show that

$I \cap L^1(K\backslash G/K) = J$. Indeed, since $L^1(K\backslash G/K)$ is a $*$-subalgebra of $L^1(G)$, if I was $*$-closed and $I \cap L^1(K\backslash G/K) = J$, then J must be $*$-closed too, which would be a contradiction.

To see that $I \cap L^1(K\backslash G/K) = J$, let $\mathbb{1}_K$ denote the characteristic function on K. We know that $L^1(K\backslash G/K) = \mathbb{1}_K * L^1(G) * \mathbb{1}_K$. Now suppose that $f \in L^1(K\backslash G/K)$. Then $f = \mathbb{1}_K * f * \mathbb{1}_K$ and so $\pi_{z,p}(f) = \pi_{z,p}(\mathbb{1}_K)\pi_{z,p}(f)\pi_{z,p}(\mathbb{1}_K)$. Note that the operator $\pi_{z,p}(\mathbb{1}_K)$ is the projection onto the span of $\mathbb{1}_{G/P}$ in $L^p(G/P)$, thus, one sees that $\pi_{z,p}(f) = 0$ if and only if $\pi_{z,p}(f)\mathbb{1}_K = 0$.

Since $\pi_{z,p}(f) = \pi_{z,p}(\mathbb{1}_K)\pi_{z,p}(f)\pi_{z,p}(\mathbb{1}_K)$, there must exist $\lambda(f) \in \mathbb{C}$ such that $\pi_{z,p}(f)\mathbb{1}_{G/P} = \lambda(f)\mathbb{1}_{G/P}$. But using the matrix coefficient as defined in Definition 2.16, and viewing $\mathbb{1}_{G/P}$ simultaneously as an element of $L^p(G/P)$ and $L^p(G/P)^* \cong L^q(G/P)$ (where q is the Hölder conjugate of p), we have that

$$\begin{aligned}\lambda(f) = \langle \pi_{z,p}(f)\mathbb{1}_{G/P}, \mathbb{1}_{G/P}\rangle &= \int_G f(x)\langle \pi_{z,p}(x)\mathbb{1}_{G/P}, \mathbb{1}_{G/P}\rangle \,\mathrm{d}x \\ &= \int_G f(x)\langle \pi_{z,2}(x)\mathbb{1}_{G/P}, \mathbb{1}_{G/P}\rangle_{L^2(G/P)} \,\mathrm{d}x = \chi_{\varphi_z}(f).\end{aligned}$$

Thus $\pi_{z,p}(f) = 0$ if and only if $\chi_{\varphi_z}(f) = 0$. This implies that $I \cap L^1(K\backslash G/K) = J$ which completes the proof. □

2.4. Proof of Theorem B

In this section we complete the proof of Theorem B from the introduction.

2.4.1. Proof of Theorem B(i).

We now give a proof of Theorem B(i). To start, we introduce some preliminary notation and results from the literature that will be used in the proof.

Let d_1 and d_2 be two natural numbers $\geqslant 2$ and denote by T_{d_1,d_2} the semi-regular tree of degree (d_1, d_2). In particular, T_{d_1,d_2} is the infinite bipartite tree such that, with respect to the associated bipartition of the vertex set $VT_{d_1,d_2} = X \sqcup Y$, all the vertices in X have degree d_1, and all the vertices in Y have degree d_2. We refer the reader to [**51**, Chapter 1] or [**34**, Chapitre I] for further information on (semi-)regular trees and groups acting on them. Standard terminology and results from these references will be used in this section.

The following is a consequence of the results in [**108**].

Lemma 2.29. [**108**] *Let X be a connected locally finite graph with infinitely many ends and suppose that $G \leqslant \mathrm{Aut}(X)$ is a closed non-compact subgroup acting transitively on the boundary of X. Then, there exists natural numbers $d_1 \geqslant 2$ and $d_2 \geqslant 3$, such that, G admits a quotient onto a closed subgroup $H \leqslant \mathrm{Aut}(T_{d_1,d_2})$ which acts transitively on the boundary of T_{d_1,d_2} and has at most two orbits on the vertices.*

It is well known that the property of a group being Wiener is preserved under taking quotients [**115**, Theorem 11.5.4], thus, to complete the proof of Theorem B(i), it suffices to show that any group satisfying the properties of H in the lemma is not Wiener. We now proceed with proving this fact.

So for the remainder of this section we fix natural numbers $d_1 \geqslant 2$ and $d_2 \geqslant 3$, and fix a closed subgroup $G \leqslant \mathrm{Aut}(T_{d_1,d_2})$ which is not compact, acts transitively on the boundary of T_{d_1,d_2}, and has at most two orbits on the vertices. Now let v be a vertex of T_{d_1,d_2} and ∞ an end of T_{d_1,d_2}. We set K and P to be the subgroups of automorphisms of G that fix v and ∞ respectively. It is well known that (G, K) is a Gelfand pair [**133**, Lemma 3.2.12] and that P is a maximal cocompact amenable subgroup of G such that $G = KP$.

Let $(v_n)_{n\in\mathbb{N}}$ be a ray in the tree T_{d_1,d_2} which lies in the end ∞. For every $p \in P$, and n sufficiently large, there exists a $k \in \mathbb{Z}$ such that $p(v_n) = p(v_{n+k})$. We then define a function $\nu : P \to \mathbb{Q}$ given by $\nu(p) := k/2$. This can be shown to be a homomorphism and its definition does not depend on the choice of the ray $(v_n)_{n\in\mathbb{N}}$ (see [**34**, Chapitre I]). Then, following the work of [**34**, Chapter 2], one defines for every $\lambda \in \mathbb{C}$ a character χ^λ on P given by

$$\chi^\lambda(p) := (\sqrt{d_1 d_2}\lambda)^{\nu(p)}.$$

We then consider, as done again in [**34**, Chapter 2], the smoothly induced representation $\pi^\lambda := \mathrm{ind}_P^G(\chi^\lambda)$ whose representation space is given by

$$V^\lambda = \{f \in C^\infty(G) : f(gp) = \chi^\lambda(p)f(g), g \in G, p \in P\}.$$

As discussed in the preliminaries, restricting functions in V^λ to K gives rise to an isomorphism $R_K : V^\lambda \to C^\infty(K/K \cap P)$, where we consider $C^\infty(K/K \cap P)$ as a subspace of $C^\infty(K)$. On V^λ, one then defines a Hermitian form by

$$\langle f, g\rangle_K := \int_K f(k)\overline{g(k)}\,\mathrm{d}k.$$

The completion of V^λ with respect to this form is a Hilbert space denoted by $\mathcal{H}^\lambda$, and it is shown in [**34**, Proposition 2.1.2] that π^λ extends to a bounded representation on $\mathcal{H}^\lambda$, denoted by Π^λ.

Now let $\varphi : K/K \cap P \to G/P$ denote the canonical homeomorphism. It is a consequence of the results in [**34**, Section 2.2] that the map $R_K \circ \varphi^{-1} : V^\lambda \to C^\infty(G/P)$ extends to an isometric isomorphism $U : \mathcal{H}^\lambda \to L^2(G/P)$. Furthermore, it is a consequence of [**34**, Proposition 2.2.1], that if $z \in \mathbb{C}$ such that $\lambda = (d_1 d_2)^{z-\frac{1}{2}}$, then $U\Pi^\lambda(g) = \pi_z(g)U$ for all $g \in G$, where $(\pi_z, L^2(G/P))$ is the boundary representation as defined in the preliminaries of this article. In particular, Π^λ and π_z are equivalent representations for $\lambda = (d_1 d_2)^{z-\frac{1}{2}}$.

Using [**34**, Théorème 2.4.6], one deduces that π_z is topologically irreducible provided the following hold:

- $z \neq 1 + \frac{2\pi i k}{\log(d_1 d_2)}$ for $k \in \mathbb{Z}$;
- $z \neq \frac{2\pi i k}{\log(d_1 d_2)}$ for $k \in \mathbb{Z}$;
- $z \neq \frac{\log(d_1)}{\log(d_1 d_2)} + \frac{\pi i(2k+1)}{\log(d_1 d_2)}$ for $k \in \mathbb{Z}$;
- $z \neq \frac{\log(d_2)}{\log(d_1 d_2)} + \frac{\pi i(2k-1)}{\log(d_1 d_2)}$ for $k \in \mathbb{Z}$.

Now, let $\mathbb{1}_K$ denote the characteristic function on K which is contained in the space $C^\infty(K/K \cap P) \cong V^\lambda \subset \mathcal{H}^\lambda$. Then, one checks by definition of U that for all $g \in G$

$$\begin{aligned}\psi_\lambda(g) := \langle \Pi^\lambda(g)\mathbb{1}_K, \mathbb{1}_K \rangle &= \langle U^{-1}\pi_z(g)U\mathbb{1}_K, \mathbb{1}_K \rangle \\ &= \langle \pi_z(g)U\mathbb{1}_K, U\mathbb{1}_K \rangle = \langle \pi_z(g)\mathbb{1}_{G/P}, \mathbb{1}_{G/P} \rangle = \varphi_z(g)\end{aligned}$$

where we again have that $\lambda = (d_1 d_2)^{z-\frac{1}{2}}$.

By [**34**, Proposition 3.2.2(1)], if $\lambda_1, \lambda_2 \in \mathbb{C}$, $\psi_{\lambda_1} = \psi_{\lambda_2}$ if and only if $\lambda_1 = \lambda_2$ or $\lambda_1 = \lambda_2^{-1}$. This implies that, for $z_1, z_2 \in \mathbb{C}$, $\varphi_{z_1} = \varphi_{z_2}$ if and only if $z_1 = z_2 + \frac{2\pi i k}{\log(d_1 d_2)}$ or $z_1 = z_2 + 1 + \frac{2\pi i k}{\log(d_1 d_2)}$ for some $k \in \mathbb{Z}$. Then, one checks that for $z \in \mathbb{C}$, $\varphi_z^*(g) = \overline{\varphi_z(g^{-1})} = \varphi_{-\overline{z}}(g)$ for all $g \in G$. It is then easy to find $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$, $\mathrm{Re}(z) \neq \frac{\log(d_1)}{\log(d_1 d_2)}$, $\mathrm{Re}(z) \neq \frac{\log(d_2)}{\log(d_1 d_2)}$, such that $\varphi_z \neq \varphi_{-\overline{z}}$. This then implies by Proposition 2.7, for this choice of z, that φ_z is not positive definite. Furthermore, since $0 < \mathrm{Re}(z) < 1/2$, $\mathrm{Re}(z) \neq \frac{\log(d_1)}{\log(d_1 d_2)}$ and $\mathrm{Re}(z) \neq \frac{\log(d_2)}{\log(d_1 d_2)}$, this ensures that π_z is topologically irreducible. Thus, by Theorem A, G is not Wiener.

2.4.2. Proof of Theorem B(ii). Throughout this section we assume some rudimentary knowledge of the theory of reductive groups and their representation theory. We use [**31**] as the main reference. To start, we set the following notation which will be used throughout the proof:

- G will denote a split reductive algebraic group over a non-archimedean local field k with residue degree q;
- A is a maximal split torus in G, M the centraliser of A in G, $N(A)$ the normaliser of A in G, and $W := N(A)/M$ the associated Weyl group;
- $\mathcal{B}$ is the Bruhat-Tits building associated to G, $\mathcal{A}$ the fundamental apartment in $\mathcal{B}$ associated to A, x_0 a special vertex in $\mathcal{A}$, and K the maximal compact open subgroup of elements in G which fix x_0;
- P is a minimal parabolic subgroup in G such that $G = KP$ and N the unipotent radical of P so that $P = MN$.

To start the proof, first note that it is well known that (G, K) is a Gelfand pair. See for example [**31**, Corollary 4.1].

Now the group P is non-unimodular and we let Δ_P be the modular function on P. To show that G is not a Wiener group, it suffices to show, by combining Theorem A, Proposition 2.13 and Proposition 2.14, that the smooth induced representation $\sigma_z := \mathrm{ind}_P^G(\Delta_P^{-z})$ is algebraically irreducible and not unitarizable for some $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$.

Let w_ℓ be the longest word of the Weyl group W and consider the function

$$\mathbb{C} \to \mathbb{C}, z \mapsto c_{w_\ell}(w_\ell \cdot \Delta_P^z) c_{w_\ell}(\Delta_P^z) \tag{2.4.1}$$

where the function c_{w_ℓ} is as defined in [**32**, §3]. One checks by definition of the function c_{w_ℓ} that the map $z \mapsto c_{w_\ell}(w_\ell \cdot \Delta_P^z) c_{w_\ell}(\Delta_P^z)$ is meromorphic and hence can have at most countably many zeroes. It then follows from [**32**, Proposition 3.5(b)] that the representation σ_z is algebraically irreducible everywhere except at countably many points.

Thus, to find a $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$ such that σ_z is algebraically irreducible and not unitarizable, it suffices to show that there are uncountably many $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$ such that σ_z is not unitarizable. To do this, it further suffices to show that there are uncountably many $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$ such that σ_z is not Hermitian i.e. $\sigma_z^+ \ncong \sigma_z$.

Note that $\sigma_{-z}^+ = \mathrm{ind}_P^G(\Delta_P^z)^+ = \mathrm{ind}_P^G(\Delta_P^{-\overline{z}}) = \sigma_{\overline{z}}$. So we need to show that there are uncountably many $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$ such that $\mathrm{ind}_P^G(\Delta_P^z) \ncong \mathrm{ind}_P^G(\Delta_P^{-\overline{z}})$. Since the representation σ_z is spherical whenever it is irreducible, it follows from Theorem 4.2(b) and Theorem 4.3 in [**31**] that the representations $\mathrm{ind}_P^G(\Delta_P^z)$ and $\mathrm{ind}_P^G(\Delta_P^{-\overline{z}})$ are equivalent if and only if there exists $w \in W$ such that $w \cdot \Delta_P^z = \Delta_P^{-\overline{z}}$ (provided that $\sigma_z = \mathrm{ind}_P^G(\Delta_P^z)$ is irreducible).

We claim that, if there exists $w \in W$ such that $w \cdot \Delta_P^z = \Delta_P^{-\overline{z}}$, then z must be either purely imaginary or purely real. This will then imply Theorem B(ii), since it implies that there are uncountably many $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$ such that σ_z is not unitarizable and is algebraically irreducible. So to prove the claim, it is well known that $P = MN$, and since G is split, the modular function on P is given by the expression

$$\Delta_P(mn) = q^{2\rho(m \cdot x_0)}$$

where ρ is the half sum of the positive roots associated to G and q is the residue characteristic of k (see Section 3.5 and Equation 24_c in [**31**] for further information). Then, if $w \cdot \Delta_P^z = \Delta_P^{-\overline{z}}$, this implies that $2z(w \cdot \rho) = -2\overline{z}\rho$, which in turn implies that $w \cdot \rho = \frac{-\overline{z}}{z}\rho$. So $w \cdot \rho$ must be a complex multiple of ρ, and in particular, this implies that either $w \cdot \rho = \rho$ or $w \cdot \rho = -\rho$. Thus, either we have $\frac{-\overline{z}}{z} = 1$ or we have $\frac{-\overline{z}}{z} = -1$, and in particular, z is either purely imaginary or purely real. This completes the proof.

2.5. Acknowledgements

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CHAPTER 3

Weighted Orlicz $*$-algebras on locally elliptic groups

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Abstract

Let G be a locally elliptic group, (Φ, Ψ) a complementary pair of Young functions, and $\omega : G \to [1, \infty)$ a weight function on G such that the weighted Orlicz space $L^\Phi(G, \omega)$ is a Banach $*$-algebra when equipped with the convolution product and involution $f^*(x) := \overline{f(x^{-1})}$ $(f \in L^\Phi(G, \omega))$. Such a weight always exists on G and we call it an L^Φ*-weight*. We assume that $1/\omega \in L^\Psi(G)$ so that $L^\Phi(G, \omega) \subseteq L^1(G)$. This paper studies the spectral theory and primitive ideal structure of $L^\Phi(G, \omega)$. In particular, we focus on studying the Hermitian, Wiener and $*$-regularity properties on this algebra, along with some related questions on spectral synthesis. It is shown that $L^\Phi(G, \omega)$ is always quasi-Hermitian, weakly-Wiener and $*$-regular. Thus, if $L^\Phi(G, \omega)$ is Hermitian, then it is also Wiener. Although, in general, $L^\Phi(G, \omega)$ is not always Hermitian, it is known that Hermitianness of $L^1(G)$ implies Hermitianness of $L^\Phi(G, \omega)$ if ω is sub-additive. We give numerous examples of locally elliptic groups G for which $L^1(G)$ is Hermitian and sub-additive L^Φ-weights on these groups. In the weighted L^1 case, even stronger Hermitianness results are formulated.

3.1. Introduction

It is a classical question in harmonic analysis and Banach algebra theory to determine for which locally compact groups G is the Banach $*$-algebra $L^1(G)$ Hermitan/symmetric, Wiener and/or $*$-regular (*c.f.* [**88, 115**]). Such a group G is called Hermitian, Wiener or Boidol if, respectively, $L^1(G)$ is Hermitian, Wiener or $*$-regular.

Classically, these problems were studied primarily in the context of connected locally compact groups and discrete groups. Every connected locally compact group can be approximated by connected Lie groups, and as a consequence, it can be shown that a connected locally compact group is Hermitian (resp. Wiener) if and only if its approximating Lie groups are all Hermitian (resp. Wiener) [**86**, Section 4].

In the case of connected Lie groups, it is well known that every connected nilpotent Lie group is Hermitian, Wiener and Boidol, and for connected solvable Lie groups of real dimension $\leqslant 4$, there is one exceptional group, the "Poguntke group" [**115**, §12.6.27], which does not satisfy these properties. On the other hand, non-compact connected semisimple Lie groups are never Hermitian, Wiener or Boidol [**115**, Chapter 12].

For discrete groups, it is a standard fact that these groups are "weakly-Wiener", and as a consequence, Hermitianness implies the Wiener property for discrete groups. It is shown in [**86**, Section 4] that a solvable finitely-generated discrete group is Hermitian if and only if it has polynomial growth.

More general results of this form are also known: for example, nilpotent groups are always Hermitian and Wiener [**95**], Boidol groups are always amenable [**13**], and it was recently proved that quasi-Hermitian groups, and hence Hermitian groups, are amenable too [**131**].

The Hermitian, Wiener and $*$-regularity properties can be formulated more generally for any Banach $*$-algebra (see [**115**]). A more general class of Banach $*$-algebras where these properties have been frequently studied are for weighted L^1-algebras, and more recently, weighted L^p-algebras, on locally compact groups [**123, 124, 50, 45, 79, 80, 81, 82**]. In particular, if G is a locally compact group, by a weight on G, we mean a measurable function $\omega : G \to [1,\infty)$ that is bounded on compact sets, sub-multiplicative and symmetric (see Definition 3.15 for more details). When G is unimodular, these conditions on ω guarantee that the weighted space $L^1(G,\omega)$ is a Banach $*$-algebra with the convolution product and the involution coming from $L^1(G)$. For $p \in (1,\infty)$ and $q := \frac{p}{p-1}$, if ω further satisfies the property that $\omega^{-q} * \omega^{-q} \leqslant \omega^{-q}$, then the space $L^p(G,\omega)$ is a Banach $*$-algebra under convolution and the same involution as in the L^1 case. Such a weight is known to exist for any σ-compact group and $p \in (1,\infty)$ [**80**, Theorem 1.1].

Weighted algebras of these forms have been studied for many years, and in the case of abelian groups, they are well understood and have many connections with classical Fourier analysis and Banach algebra theory. See for example [**43**] and the references there in.

A celebrated result in the direction of weighted L^p-algebras is that if G is a compactly generated group of polynomial growth, and ω a weight on G satisfying some technical growth conditions, then $L^1(G,\omega)$ is Hermitian, Wiener and $*$-regular [**50, 45**]. Similar results have also been obtained for weighted L^p-algebras ($p \in (1,\infty)$) on compactly generated groups of polynomial growth [**82**], however, some difficulties in determining which of these algebras are Hermitian prevent the results from being as general as in the L^1 case. This line of work relies critically on the proof by Losert that every compactly generated group of polynomial growth is Hermitian, which is a consequence of his structure theory for such groups [**92**]. The work is also motivated by and refers to some older results of Hulanicki, Pytlik and Dixmier on the spectral theory and functional calculus of these algebras [**41, 69, 70, 123, 124**].

Even more recently, these results about weighted L^p-algebras on compactly generated groups of polynomial growth were generalised and extended in the setting of (twisted) weighted Orlicz $*$-algebras [**110, 111**]. An Orlicz space is a generalisation of an L^p-space which was defined in the 1930's by Orlicz [**109**]. Both L^p-spaces and variable Lebesgue spaces provide standard examples of Orlicz spaces, and certain Sobolev spaces can be found as subspaces of Orlicz spaces. Analysts have been interested in various classes of Orlicz spaces over the years, due to, for example, their applications in partial differential equations, calculus of variations and physics [**37, 102, 64**]. Thus, Orlicz spaces form a natural generalisation of L^p-spaces to study.

To be more explicit, one constructs Orlicz spaces as follows: given a measure space (X, μ) and a Young function $\Phi : \mathbb{R} \to [0, \infty]$, one can associate a certain Banach space $L^\Phi(X)$ of measurable complex valued functions on X, and it is this space that we call an *Orlicz space* (see Section 3.2.2 for the complete definition or [**127**] for further theoretical details). The case when $\Phi(x) = |x|^p/p$ gives the classical L^p-spaces. Now, given a locally compact group G and a Young function Φ, one can study the space $L^\Phi(G)$. Typically $L^\Phi(G)$ is not closed under convolution, even when this is an L^p-space, but if G is unimodular and ω a suitable weight on G, the weighted Orlicz space $L^\Phi(G, \omega)$ may be a Banach $*$-algebra when equipped with the convolution product and the involution $f^*(x) := \overline{f(x^{-1})}$ $(f \in L^\Phi(G, \omega))$. This is what we will refer to as a *weighted Orlicz $*$-algebra* (or sometimes a *weighted L^Φ-algebra*) and we will call such a weight ω an *L^Φ-weight*.

In this paper we study the harmonic analysis of weighted Orlicz $*$-algebras on locally elliptic groups. A locally compact group is locally elliptic if and only if it can be written as a countable ascending union of compact open subgroups. In particular, any non-compact locally elliptic group cannot be compactly generated, however, these groups have polynomial growth. This work, in particular, tests the necessity of the compactly generated assumption in the work on compactly generated groups of polynomial growth described in the previous paragraphs. Furthermore, every (not necessarily compactly generated) locally compact group of polynomial growth is the extension of a locally elliptic group by a Lie group [**93**, Theorem 3.3], so understanding weighted Orlicz $*$-algebras on general groups of polynomial growth would naturally require one to understand what happens in the locally elliptic case.

Locally elliptic groups also feature in many places throughout the theory of totally disconnected locally compact (tdlc) groups and they thus form an interesting class of groups to study the harmonic analysis of from the perspective of tdlc group theory. For example, locally elliptic groups have strong connections with the theory of contraction groups [**6, 58, 59, 60**] and the theory of scale groups [**6, 66, 152**]. These two classes of groups are actively studied in tdlc group theory and play an important role. Also, any unipotent linear algebraic group over a non-archimedean local field is locally elliptic, so there is potential, as a consequence of the results in this paper, to construct an analogous theory of spectral synthesis and weighted algebras on such groups as there is in the case of connected nilpotent Lie groups [**97, 98, 99, 94, 9**].

The motivation and goals of the present paper are the following:

(i) To show that many of the arguments and results for weighted Orlicz $*$-algebras on compactly generated groups of polynomial growth hold equally well for locally elliptic groups. In certain aspects, the theory in the locally elliptic case is even cleaner than the compactly generated case.
(ii) Give new examples of weighted Orlicz $*$-algebras that have nice Banach algebra properties, such as being (quasi-)Hermitian, (weakly-)Wiener and $*$-regular, and provide a range of examples of groups and weights that fit into this theory.
(iii) Initiate further research into understanding the harmonic analysis of tdlc groups and present the work in a way that is relatively accessible to researchers in tdlc group theory.

As a consequence of the above motives, particular (iii), we choose to be more elaborate in our exposition so that the paper is accessible to researchers who are not experts in the theory of Banach algebras nor familiar with this line of research. In particular, we note well that a number of the arguments in this paper are already well known and written down in the literature, but we choose to include them here for completeness and expository purposes.

After going through some preliminaries and introductory material on Banach $*$-algebras, Orlicz spaces and locally elliptic groups in Section 5.2, we study the properties of weights on locally elliptic groups in Section 5.3, which is critical to our later results. In particular, in Proposition 3.25, we show that every weight on a locally elliptic group satisfies the GRS condition, which is an important condition used for obtaining Hermitianness of weighted L^1-algebras in the context of compactly generated groups of polynomial growth [**49**]. We also show that every weight ω on a locally elliptic group is dominated by a sub-additive weight $\omega_1^\sharp$ such that $1/\omega_1^\sharp \in L^1(G) \cap L^\infty(G)$, see Proposition 3.28. Since $\omega_1^\sharp$ is sub-additive and

$1/\omega_1^\sharp \in L^1(G) \cap L^\infty(G)$, it can be shown that $L^\Phi(G,\omega_1^\sharp)$ is a Banach $*$-algebra for any Young function Φ [**111**, Theorem 4.5]. In particular, every weight on a locally elliptic group is dominated by a sub-additive L^Φ-weight, for any given Young function Φ. This later property is used extensively throughout the article.

To construct the weight $\omega_1^\sharp$ as in Proposition 3.28, we need to write the locally elliptic group G as an ascending union of compact open subgroups $G = \bigcup_{i=1}^\infty K_i$ such that the sequence of indices $([K_{i+1} : K_i])_{i=1}^\infty$ is non-decreasing. Such a set of compact open subgroups $(K_i)_{i\in\mathbb{N}}$ satisfying the property that the sequence of indices $([K_{i+1} : K_i])_{i=1}^\infty$ is non-decreasing will be called a ***standard decomposition*** of G. It is obvious that any locally elliptic group has a standard decomposition.

The following result is a consequence of the above results on weights. We note that the assumption in the following theorem that $1/\omega \in L^\Psi(G)$ implies $L^\Phi(G,\omega) \subseteq L^1(G)$ by the Hölder inequality for Orlicz spaces [**127**, Section 3.3]. Also, we use $\nu_\mathcal{A}$ to denote the spectral radius function on a Banach algebra $\mathcal{A}$.

Theorem 3.1. *Let G be a locally elliptic group, $(K_i)_{i\in\mathbb{N}}$ a standard decomposition of G, (Φ,Ψ) a complementary pair of Young functions and ω an L^Φ-weight on G with $1/\omega \in L^\Psi(G)$. Define $\omega_1^\sharp$ to be the weight constructed in Proposition 3.28 with respect to the compact open subgroups $(K_i)_{i\in\mathbb{N}}$. Then, for all $f \in L^\Phi(G,\omega_1^\sharp) \subseteq L^\Phi(G,\omega)$, $\nu_{L^\Phi(G,\omega)}(f) = \nu_{L^1(G)}(f)$.*

As already mentioned earlier, as a consequence of Losert's work [**92**], if G is a compactly generated group of polynomial growth, $L^1(G)$ is always Hermitian. Then, if ω is a weight on G, $L^1(G,\omega)$ is Hermitian if and only if ω satisfies the GRS condition [**49**, Theorem 1.3]. In contrast, if we now let G be a locally elliptic group, $L^1(G)$ is not always Hermitian (see Section 3.4), however, it is quasi-Hermitian as a consequence of [**131**, Remark 4.10]. Part (i) of the following theorem is then a consequence of Theorem 3.1, but it can also been proved via the fact that every weight on a locally elliptic group satisfies the GRS condition. Part (ii) follows from [**111**, Theorem 4.5]. The proof of Theorem 3.2 is found in Section 3.4. In the following, we use $\sigma_\mathcal{A}(x)$ to denote the spectrum of an element x in a Banach algebra $\mathcal{A}$.

Theorem 3.2. *Let G be a locally elliptic group, (Φ,Ψ) a complementary pair of Young functions and ω an L^Φ-weight on G with $1/\omega \in L^\Psi(G)$. The following hold:*

- *(i) The Banach $*$-algebra $L^\Phi(G,\omega)$ is quasi-Hermitian i.e. for every self-adjoint function $f \in C_c(G) \subseteq L^\Phi(G,\omega)$, $\sigma_{L^\Phi(G,\omega)}(f) \subseteq \mathbb{R}$;*
- *(ii) If G is Hermitian and ω sub-additive, then for all $f \in L^\Phi(G,\omega)$, $\sigma_{L^\Phi(G,\omega)}(f) = \sigma_{L^1(G)}(f)$. In particular, $L^\Phi(G,\omega)$ is Hermitian.*

Although a locally elliptic group G is not always Hermitian, one can always construct a weight ω on G such that $L^1(G, \omega)$ is Hermitian (see Section 5.3 and Section 3.4). Also, we give a number of examples of Hermitian locally elliptic groups and sub-additive L^Φ-weights on these groups in Section 3.8.

In Section 3.5, we prove that every weighted Orlicz $*$-algebra on a locally elliptic group is weakly-Wiener. The proof requires one to construct a functional calculus for certain smooth periodic functions. This functional calculus is also constructed in Section 3.5 and makes use of our results about weights on locally elliptic groups.

Theorem 3.3. *Let G be a locally elliptic group, (Φ, Ψ) a complementary pair of Young funcitons and ω an L^Φ-weight on G such that $1/\omega \in L^\Psi(G)$. The Banach $*$-algebra $L^\Phi(G, \omega)$ is weakly-Wiener. In particular, if $L^\Phi(G, \omega)$ is Hermitian, then it is Wiener.*

Specialising the above theorem to the case of $L^1(G)$, we also see that every locally elliptic group is weakly-Wiener. We note this was already known since [**95**], but we list it here to be explicit.

Corollary 3.4. *Every locally elliptic group is weakly-Wiener. In particular, every Hermitian locally elliptic group is Wiener.*

Section 3.6 studies the representation theory, $*$-regularity property and C^*-enveloping algebras of a weighted Orlicz $*$-algebra on a locally elliptic group. The main result is the following theorem. In the statement of the theorem, the condition that $\Phi \in \Delta_2$ is defined in Definition 3.12, and this assumption, for example, guarantees that $L^\Phi(G, \omega)^* = L^\Psi(G, \omega^{-1})$, where Ψ is the complementary Young function to Φ.

Theorem 3.5. *Let G be a locally elliptic group, (Φ, Ψ) a complementary pair of Young functions with $\Phi \in \Delta_2$, and ω an L^Φ-weight on G such that $1/\omega \in L^\Psi(G)$. The following hold:*

- *(i)* $C^*(L^\Phi(G, \omega)) \cong C^*(G)$;
- *(ii) The algebra* $L^\Phi(G, \omega)$ *is $*$-regular, in particular,* $\mathrm{Prim}_*(L^\Phi(G, \omega))$, $\mathrm{Prim}_*(L^1(G))$ *and* $\mathrm{Prim}(C^*(G))$ *are homeomorphic.*

This result could be useful in the study of the unitary representation theory of locally elliptic groups. Indeed, for example, it is currently an open question for which locally elliptic groups G does the topology on $\mathrm{Prim}(C^*(G))$ satisfy the T_1 separation axiom. Sometimes $C^*(G)$ is not the most convenient algebra to work with: being a completion of $L^1(G)$, one cannot treat all elements of $C^*(G)$ as functions on G. Thus, it is often convenient to work with $L^1(G)$ instead. In the case of a locally elliptic group, as given by the above theorem, $\mathrm{Prim}_*(L^1(G))$ and

$\mathrm{Prim}(C^*(G))$ are homeomorphic, so it is fine working with $L^1(G)$ instead of $C^*(G)$ in the study of the unitary representation theory. However, unlike $C^*(G)$, $L^1(G)$ may not be Hermitian, which is a desirable property to have. But even if $L^1(G)$ is not Hermitian, as is shown in the article, there always exist weights on G such that $L^1(G,\omega)$ is Hermitian (and $\mathrm{Prim}_*(L^1(G,\omega)) \cong \mathrm{Prim}(C^*(G))$). So in the context of non-Hermitian locally elliptic groups, working with a weighted L^1-algebra may be more natural algebra than working with $C^*(G)$ or $L^1(G)$ when studying the unitary representation theory of these groups.

In Section 3.7, we show that every hull of a weighted Orlicz $*$-algebra contains a minimal ideal. This leads to interesting questions to pursue in the context of spectral synthesis, which we discuss in Section 3.8. In Section 3.8, we also go through a variety of examples of locally elliptic groups and weighted algebras on these groups. Furthermore, we pose some open questions that could be investigated in future work.

3.2. Preliminaries

3.2.1. Banach $*$-algebras and representation theory.

We begin the preliminaries section by collecting some notation, definitions and results on Banach algebras that will be used throughout the article. We assume the reader has some familiarity with (Banach) $*$-algebras, as can be found in [**42, 106**], for example.

Notation 3.6. Let $\mathcal{A}$ be a Banach $*$-algebra and $x \in \mathcal{A}$.

(i) $\|x\|_{\mathcal{A}}$ denotes the ***norm*** of x in $\mathcal{A}$.
(ii) $\sigma_{\mathcal{A}}(x)$ denotes the ***spectrum*** of x in $\mathcal{A}$.
(iii) $\nu_{\mathcal{A}}(x)$ denotes the ***spectral radius*** of x in $\mathcal{A}$.

We now define some notation and terminology in regards to the representation theory of Banach $*$-algebras.

Notation 3.7. Let $\mathcal{A}$ be a Banach $*$-algebra.

(i) A ***unitary representation*** of $\mathcal{A}$ is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a non-degenerate $*$-homomorphism to the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$.
(ii) $\widehat{\mathcal{A}}$ denotes the set of all topologically irreducible unitary representations of $\mathcal{A}$ upto unitary equivalence equipped with the Fell topology.
(iii) $\mathrm{Prim}_*(\mathcal{A})$ denotes the space of kernels of topological irreducible unitary representations of $\mathcal{A}$ equipped with the hull-kernel topology.
(iv) $\mathrm{Prim}(\mathcal{A})$ denotes the space of annihilators of simple $\mathcal{A}$-modules equipped with the hull-kernel topology.

We will now remind the reader of the definition of the hull-kernel topology and Fell topology: given an arbitrary subset $X \subseteq \mathcal{A}$, define $\mathrm{hull}_*(X) := \{I \in \mathrm{Prim}_*(\mathcal{A}) : X \subseteq I\}$ (resp. $\mathrm{hull}(X) := \{I \in \mathrm{Prim}(\mathcal{A}) : X \subseteq I\}$). Then, the ***hull-kernel topology*** on $\mathrm{Prim}_*(\mathcal{A})$ (resp. $\mathrm{Prim}(\mathcal{A})$) is the topology generated by defining the sets $X \subseteq \mathrm{Prim}_*(\mathcal{A})$ (resp. $X \subseteq \mathrm{Prim}(\mathcal{A})$) satisfying $\mathrm{hull}_*(\ker(X)) = X$ (resp. $\mathrm{hull}(\ker(X)) = X$) to be closed, where $\ker(X) = \bigcap_{I \in X} I$. The Fell topology on $\widehat{\mathcal{A}}$ is precisely the pullback of the hull-kernel topology on $\mathrm{Prim}_*(\mathcal{A})$ via the canonical surjection $\widehat{\mathcal{A}} \twoheadrightarrow \mathrm{Prim}_*(\mathcal{A})$.

The notion of a C^*-enveloping algebra is critical to understanding the $*$-regularity property.

Definition 3.8. Let $\mathcal{A}$ be a Banach $*$-algebra.

(i) The ***maximal C^*-norm*** on $\mathcal{A}$ is the norm defined on $x \in \mathcal{A}$ by

$$\|x\|_{\max} := \sup\{\|\pi(x)\|_{\mathcal{B}(\mathcal{H}_\pi)} : \pi \in \widehat{\mathcal{A}}\}.$$

(ii) The ***reducing ideal*** of $\mathcal{A}$, denoted $\mathcal{A}_R$, is the $*$-ideal in $\mathcal{A}$ consisting of all elements $x \in \mathcal{A}$ with $\|x\|_{\max} = 0$.

(iii) The ***enveloping C^*-algebra*** of $\mathcal{A}$, denoted $C^*(\mathcal{A})$, is the completion of $\mathcal{A}/\mathcal{A}_R$ with respect to the maximal C^*-norm.

The following proposition is fundamental to understanding the Hermitian property for a Banach $*$-algebra. We recall that an element $x \in \mathcal{A}$ of a Banach $*$-algebra $\mathcal{A}$ is ***self-adjoint*** if $x = x^*$.

Proposition 3.9. [**87**] *Let $\mathcal{A}$ be a Banach $*$-algebra. The following are equivalent:*

(i) For all $x \in \mathcal{A}$, $\sigma_{\mathcal{A}}(x^*x) \subseteq \mathbb{R}_{\geqslant 0}$;

(ii) For all self-adjoint $x \in \mathcal{A}$, $\sigma_{\mathcal{A}}(x) \subseteq \mathbb{R}$;

(iii) $\mathrm{Prim}(\mathcal{A}) \subseteq \mathrm{Prim}_*(\mathcal{A})$ i.e. every simple $\mathcal{A}$-module is unitarizable.

We now define the Hermitian, Wiener and $*$-regularity properties that this article focuses on studying.

Definition 3.10. Let $\mathcal{A}$ be a Banach $*$-algebra.

(i) $\mathcal{A}$ is called ***Hermitian*** if one of the equivalent conditions of Proposition 3.9 hold.

(ii) $\mathcal{A}$ is called ***Wiener*** if for every proper closed two-sided ideal $I \subseteq \mathcal{A}$, there exists $J \in \mathrm{Prim}_*(\mathcal{A})$ such that $I \subseteq J$.

(iii) $\mathcal{A}$ is called ***weakly-Wiener*** if for every proper closed two-sided ideal $I \subseteq \mathcal{A}$, there exists $J \in \mathrm{Prim}(\mathcal{A})$ such that $I \subseteq J$.

(iv) $\mathcal{A}$ is called ***$*$-regular*** if $\mathrm{Prim}_*(\mathcal{A})$ and $\mathrm{Prim}(C^*(\mathcal{A})) = \mathrm{Prim}_*(C^*(\mathcal{A}))$ are homeomorphic.

3.2.2. Orlicz spaces. In this subsection we will introduce basic concepts concerning Orlicz spaces and Orlicz $*$-algebras, and give some results that will be used throughout the article. We primarily follow the references [**127, 110, 111**] and the reader can consult these for further details on the topic.

Definition 3.11. A ***Young function*** is a function $\Phi : \mathbb{R} \to [0, \infty]$ which satisfies the following conditions:

(i) Φ is convex;
(ii) Φ is even;
(iii) $\Phi(0) = 0$;
(iv) $\lim_{x\to\infty} \Phi(x) = +\infty$.

One associates to the Young function Φ its ***complementary Young function*** defined by

$$\Psi(y) := \sup\{x|y| - \Phi(x) : x \geqslant 0\}.$$

It can be checked that Ψ is also a Young function and the complementary Young function to Ψ is Φ. We call (Φ, Ψ) a ***complementary pair*** of Young functions.

A complementary pair of Young functions (Φ, Ψ) satisfies Young's inequality:

$$xy \leqslant \Phi(x) + \Psi(y) \quad \forall x, y \in \mathbb{R}.$$

The following notion of the Δ_2-condition for a Young function will be important later. We will mention later some consequences of this condition.

Definition 3.12. Let Φ be a Young function. We say that Φ satisfies the ***Δ_2-condition***, or $\Phi \in \Delta_2$, if there exists a constant $C > 0$ such that

$$\Phi(2x) \leqslant C\Phi(x)$$

for all $x \geqslant 0$.

We now define the Orlicz space associated to a Young function. For the remainder of this subsection, unless otherwise stated, G will be a locally compact group and all integration is performed against some prior fixed left-Haar measure on G.

Definition 3.13. Let (Φ, Ψ) be a complementary pair of Young functions. The ***Orlicz space*** on G associated to Φ is the space

$$L^\Phi(G) := \left\{ f : G \to \mathbb{C} \text{ measurable} : \int_G \Phi(\alpha|f|)\, dx < \infty \text{ for some } \alpha > 0 \right\}.$$

The space $L^\Phi(G)$ is equipped with the ***Orlicz norm***

$$\|f\|_{L^\Phi(G)} := \sup\left\{ \int_G |f(x)g(x)|\, dx : \int_G \Psi(|g(x)|)\, dx \leqslant 1 \right\} \quad (f \in L^\Phi(G))$$

with respect to which $L^{\Phi}(G)$ becomes a Banach space. The Orlicz norm is equivalent to the ***Luxemburg norm*** which is defined as

$$N_{\Phi}(f) := \inf\left\{k > 0 : \int_G \Phi(|f(x)|/k)\,dx \leqslant 1\right\} \quad (f \in L^{\Phi}(G)).$$

We note that in the case that $\Phi(x) = |x|^p/p$ $(p \in (1,\infty))$, the complementary Young function to Φ is $\Psi = |x|^q/q$ where $1/p + 1/q = 1$, and these Young functions satisfy the Δ_2 condition. Furthermore, in this case, $L^{\Phi}(G)$ becomes the classical L^p-space denoted $L^p(G)$, and the Orlicz norm with respect to the Young function Φ is equivalent to the usual L^p-norm on $L^p(G)$.

The following provides some other examples of Young functions, some of which arise in physics and probablilty theory.

Example 3.14. [**102**] The following are Young functions:

(i) $\Phi = |x|^p/p$ $(p \in [1,\infty))$;
(ii) $\Phi(x) = e^{|x|} - |x| - 1$;
(iii) $\Phi(x) = \cosh(x) - 1$;
(iv) $\Phi(x) = x\log(1+x)$.

We now move on to studying weights and weighted Orlicz spaces.

Definition 3.15. A ***weight*** ω on a locally compact group G is a measurable function $\omega : G \to [1,\infty)$ that satisfies the following properties:

(i) ω is bounded on compact sets;
(ii) ω is ***sub-multiplicative*** i.e. $\omega(xy) \leqslant \omega(x)\omega(y)$ for all $x, y \in G$;
(iii) ω is ***symmetric*** i.e. $\omega(x) = \omega(x^{-1})$ for all $x \in G$.

Two weights ω and ω' on G are ***equivalent*** if there exists constants C and C' such that $C\omega(x) \leqslant \omega'(x) \leqslant C'\omega(x)$ for all $x \in G$.

The notion of a sub-additive weight will be used throughout the article. A sub-additive weight is also often referred to as a polynomial weight, which was defined by Pytlik in [**124**].

Definition 3.16. Let ω a weight on G. The weight ω is called ***sub-additive*** if there exists a constant $C > 0$ such that for all $x, y \in G$

$$\omega(xy) \leqslant C(\omega(x) + \omega(y)).$$

We now define the notion of a weighted Orlicz space.

Definition 3.17. Let Φ be a Young function and ω a weight on G. The ***weighted Orlicz space*** on G corresponding to Φ and ω is the space

$$L^{\Phi}(G,\omega) := \{f \in L^{\Phi}(G) : \|f\omega\|_{L^{\Phi}(G)} < \infty\}$$

which becomes a Banach space when equipped with the norm

$$\|f\|_{L^\Phi(G,\omega)} := \|f\omega\|_{L^\Phi(G)} \quad (f \in L^\Phi(G,\omega)).$$

We note that if we have two equivalent weights ω and ω' on G, then the corresponding weighted Orlicz spaces $L^\Phi(G,\omega)$ and $L^\Phi(G,\omega')$ are isomorphic as Banach spaces. Also, if $\Phi \in \Delta_2$, then one can show that the dual of $L^\Phi(G,\omega)$ is $L^\Phi(G,\omega)^* = L^\Psi(G,\omega^{-1})$, where Ψ is the complementary Young function to Φ. In particular, if Ψ is also Δ_2, then $L^\Phi(G,\omega)$ is a reflexive Banach space.

The following elementary fact will be used throughout the article.

Proposition 3.18. *Let G be a locally compact group, (Φ,Ψ) a complementary pair of Young functions, and ω a weight on G. If $1/\omega \in L^\Psi(G)$ then $L^\Phi(G,\omega) \subseteq L^1(G)$.*

Proof. The proof follows directly from the Hölder inequality for Orlicz spaces [**127**, Section 3.3]. □

The following result is critical to our article.

Proposition 3.19. [**111**, Theorem 4.5] *Let G be a unimodular locally compact group, (Φ,Ψ) a complementary pair of Young functions and ω a weight on G. If ω is sub-additive and $1/\omega \in L^\Psi(G)$, then $L^\Phi(G,\omega) \subseteq L^1(G)$ is a Banach $*$-algebra when equipped with the convolution product and the involution $f^*(x) := \overline{f(x^{-1})}$ $(f \in L^\Phi(G,\omega))$. Moreover, the following hold:*

(i) There exists a constant $C > 0$ such that for all $f, g \in L^\Phi(G,\omega)$

$$\|f * g\|_{L^\Phi(G,\omega)} \leqslant C(\|f\|_{L^1(G)}\|g\|_{L^\Phi(G,\omega)} + \|f\|_{L^\Phi(G,\omega)}\|g\|_{L^1(G)});$$

(ii) If $L^1(G)$ is Hermitian, then $L^\Phi(G,\omega)$ is Hermitian.

It can also be the case that $L^\Phi(G,\omega)$ becomes a Banach $*$-algebra under convolution even when ω is not sub-additive. We make the following definition for ease of terminology throughout this paper.

Definition 3.20. Let Φ be a Young function and ω a weight on a unimodular locally compact group G. We call ω an ***L^Φ-weight*** if $L^\Phi(G,\omega)$ is a Banach $*$-algebra when equipped with the convolution product and the involution $f^*(x) = \overline{f(x^{-1})}$.

To finish the section on Orlicz spaces, we state some standard facts about Orlicz algebras that will be used in the article. The proofs can be found in the papers [**110, 111**].

Proposition 3.21. [**110, 111**] *Let G be a unimodular locally compact group with left Haar measure μ, (Φ,Ψ) a complementary pair of Young functions and ω a L^Φ-weight on G. The following hold:*

(i) *For* $x \in G$, *let* L_x *be the operator on* $L^\Phi(G,\omega)$ *defined by* $L_x(f)(y) := f(x^{-1}y)$ *for* $f \in L^\Phi(G,\omega)$. *Then,* $\|L_x f\|_{L^\Phi(G,\omega)} \leqslant \omega(x)\,\|f\|_{L^\Phi(G,\omega)}$ *for all* $f \in L^\Phi(G,\omega)$.

(ii) *Let* $\mathcal{N}$ *be the set of compact neighbourhoods of the identity in* G. *Then,* $(\chi_K/\mu(K))_{K\in\mathcal{N}}$, *where* χ_K *denotes the characteristic function on the set* K, *is an approximate identity in* $L^\Phi(G,\omega)$. *It is a bounded approximate identity in* $L^1(G,\omega)$.

(iii) *The space* $L^\Phi(G,\omega)$ *is a* $L^1(G,\omega)$*-module with respect to convolution. For* $f \in L^1(G,\omega)$ *and* $g \in L^\Phi(G,\omega)$, *we have that*

$$\|f * g\|_{L^\Phi(G,\omega)} \leqslant \|f\|_{L^1(G,\omega)}\,\|g\|_{L^\Phi(G,\omega)}\,.$$

3.2.3. Locally elliptic locally compact groups.

First we define the notion of a locally elliptic group.

Definition 3.22. Let G be a locally compact group. The group G is called ***locally elliptic*** if every compact subset of G generates a relatively compact subgroup.

Of course, if G is discrete, then locally elliptic is synonymous with the term locally finite. One also checks easily that every locally elliptic group is unimodular.

The following result, which is a consequence of work of Platonov, gives an equivalent characterisation of locally elliptic groups.

THEOREM 3.23. [**117**] *Let* G *be a locally compact group. Then* G *is locally elliptic if and only if it is a countable increasing union of compact open subgroups.*

Throughout the remainder of the article, if we say that $G = \bigcup_{i=1}^{\infty} K_i$ is a locally elliptic group, then we implicitly assume that G is locally compact, the K_i are compact open subgroups of G, and $K_i \leqslant K_{i+1}$ for all $i \in \mathbb{N}$. Furthermore, we assume that G is not compact to avoid any trivialities.

We give numerous examples of locally elliptic groups and their connection with the theory in this paper in Section 3.8.

3.3. Weights on locally elliptic groups

In this section we will study some properties concerning weights on locally elliptic groups that will be critical to our results later in the article.

In the context of weighted L^1-algebras on compactly generated groups of polynomial growth, the condition of a weight satisfying the GRS condition is an important part of the theory. Indeed, a weighted L^1-algebra $L^1(G,\omega)$ on a compactly generated group of polynomial growth G is Hermitian if and only if ω satisfies the GRS condition [**49**, Theorem 1.3].

Definition 3.24. Let G be a locally compact group and ω a weight on G. The weight ω is said to satisfy the ***GRS condition***, if for all $x \in G$, $\lim_{n\to\infty} \omega(x^n)^{1/n} = 1$.

We will now show that every weight on a locally elliptic group satisfies the GRS condition.

Proposition 3.25. *Let G be a locally elliptic group and ω a weight on G. Then ω satisfies the GRS condition.*

Proof. Let $x \in G$. Since G is locally elliptic, the set $\{x^n : n \in \mathbb{N}\}$ is contained in some compact open subgroup $K \leqslant G$. By definition, ω is bounded on compact sets, so $C := \sup\{\omega(k) : k \in K\}$ is finite. Then, for all $n \in \mathbb{N}$, $\omega(x^n) \leqslant C$. Thus we have that

$$\lim_{n\to\infty} \omega(x^n)^{1/n} \leqslant \lim_{n\to\infty} C^{1/n} = 1.$$

□

We now define a class of weights that exist on a locally elliptic group. These weights will be important throughout the article.

Definition 3.26. Let $G = \bigcup_{i=1}^{\infty} K_i$ be a locally elliptic group and $\mathbf{a} := (a_i)_{i=1}^{\infty} \subseteq \mathbb{R}_{\geqslant 1}$ a sequence. Define a function $\omega_{\mathbf{a}}$ on G by

$$\omega_{\mathbf{a}} := a_1 \chi_{K_1} + \sum_{i=2}^{\infty} a_i \chi_{K_i \setminus K_{i-1}}$$

where χ_U denotes the characteristic function on the set $U \subseteq G$. If $\mathbf{a}$ is a non-decreasing sequence, one checks easily that $\omega_{\mathbf{a}}(xy) \leqslant \max\{\omega_{\mathbf{a}}(x), \omega_{\mathbf{a}}(y)\}$ for all $x, y \in G$, from which it follows that $\omega_{\mathbf{a}}$ is sub-additive weight on G.

We now show that any weight on a locally elliptic group G is dominated by a weight of the form $\omega_{\mathbf{a}}$. Furthermore, we show that $\mathbf{a}$ can be chosen so that $\omega_{\mathbf{a}}$ is an L^{Φ}-weight for any Young function Φ. We use the following lemma in the proof of the next proposition.

Lemma 3.27. [**47**, Korollar 3.8] *Let G be a unimodular locally compact group and ω a locally-integrable sub-additive weight on G. If $p \in (1, \infty)$ and $\omega^{-1} \in L^q(G)$ where $q := \frac{p}{p-1}$, then $L^p(G, \omega)$ is a Banach $*$-algebra.*

We note that there is a mistake in the statement of [**47**, Korollar 3.8] but we have corrected it in our version. Also, in [**47**], weights are not assumed to be symmetric, so the conclusion in [**47**, Korollar 3.8] is that $L^p(G, \omega)$ is a Banach algebra, not a Banach $*$-algebra.

Proposition 3.28. *Let* $G = \bigcup_{i=1}^{\infty} K_i$ *be a locally elliptic group,* μ *a left Haar measure on* G *normalised so that* $\mu(K_1) = 1$*,* $p \in [1,\infty)$ *and* ω *a weight on* G*. Set* $a_i := \sup\{\omega(x) : x \in K_i\}$ *for each* $i \in \mathbb{N}$*. Define two functions on* G *by*

$$\omega^{\sharp} := a_1\chi_{K_1} + \sum_{i=2}^{\infty} a_i\chi_{K_i\backslash K_{i-1}}$$

and

$$\omega_p^{\sharp} := (a_1+1)\chi_{K_1} + \sum_{i=2}^{\infty}(a_i+i^2)\mu(K_i\backslash K_{i-1})^{1/q}\chi_{K_i\backslash K_{i-1}}$$

where $q = \frac{p}{p-1}$ *if* $p > 1$*, and* $q = 1$ *if* $p = 1$*. Then,* $\omega^{\sharp}$ *is a sub-additive weight on* G*,* $\omega \leqslant \omega^{\sharp} \leqslant \omega_p^{\sharp}$*, and* $1/\omega_p^{\sharp} \in L^q(G)$*. Furthermore, if the sequence of indices* $([K_{i+1} : K_i])_{i=1}^{\infty}$ *is non-decreasing, then* $\omega_p^{\sharp}$ *is a sub-additive* L^p*-weight on* G*.*

Proof. Since each of the K_i are compact open subgroups of G and ω is bounded on compact sets by definition, it follows that $a_i := \sup\{\omega(x) : x \in K_i\}$ is finite for each i. Also, since $K_i \leqslant K_{i+1}$ for each i, the sequence $(a_i)_{i=1}^{\infty} \subseteq \mathbb{R}_{\geqslant 1}$ is non-decreasing, and hence $\omega^{\sharp}$ is a sub-additive weight on G by the prior discussion. Similarly, if the sequence $([K_{i+1} : K_i])_{i=1}^{\infty}$ is non-decreasing, then the sequence $((a_i + i^2)\mu(K_i\backslash K_{i-1})^{1/q})_{i=1}^{\infty}$ (where we take K_0 to be the empty set) is also non-decreasing, and hence $\omega_p^{\sharp}$ is a sub-additive weight on G.

Since we assume that K_i is a proper subgroup of K_{i+1} for each i, it follows that

$$\begin{aligned}\mu(K_{i+1}\backslash K_i) &= \mu(K_{i+1}) - \mu(K_i)\\ &= [K_{i+1} : K_i]\mu(K_i) - \mu(K_i)\\ &\geqslant 2\mu(K_i) - \mu(K_i) \geqslant 1.\end{aligned}$$

It is then clear, by definition of $\omega^{\sharp}$ and $\omega_p^{\sharp}$, that $\omega \leqslant \omega^{\sharp} \leqslant \omega_p^{\sharp}$.

We now just need to show that $\omega_p^{\sharp}$ is an L^p-weight. First, we show that $1/\omega_p^{\sharp} \in L^q(G)$. To do this, one checks that

$$1/\omega_p^{\sharp} = \frac{\chi_{K_1}}{a_1+1} + \sum_{i=2}^{\infty}\frac{\chi_{K_i\backslash K_{i-1}}}{(a_i+i^2)\mu(K_i\backslash K_{i-1})^{1/q}}$$

from which it follows that for any $q \in [1,\infty)$

$$\begin{aligned}\int_G (1/\omega_p^{\sharp}(x))^q\, d\mu(x) &= \int_{K_1}(1/\omega_p^{\sharp}(x))^q\, d\mu(x) + \sum_{i=2}^{\infty}\int_{K_i\backslash K_{i-1}}(1/\omega_p^{\sharp}(x))^q\, d\mu(x)\\ &= \frac{1}{(a_1+1)^q} + \sum_{i=2}^{\infty}\frac{1}{(a_i+i^2)^q}\\ &\leqslant \frac{1}{(a_1+1)^q} + \sum_{i=2}^{\infty}\frac{1}{i^{2q}} < \infty.\end{aligned}$$

Thus $1/\omega_p^\sharp \in L^q(G)$. It then follows by Lemma 3.27 that $\omega_p^\sharp$ is an L^p-weight on G. This completes the proof. $\square$

We recall from the introduction that if G is locally elliptic group, a ***standard decomposition*** of G is a set of compact open subgroups $(K_i)_{i\in\mathbb{N}}$ of G such that $K_i \leqslant K_{i+1}$ for each i, $G = \bigcup_{i=1}^{\infty} K_i$ and the sequence of indices $([K_{i+1} : K_i])_{i=1}^{\infty}$ is non-decreasing. The following is then a direct corollary of the theorem.

Corollary 3.29. *Let G be a locally elliptic group, $(K_i)_{i\in\mathbb{N}}$ a standard decomposition of G, (Φ, Ψ) a complementary pair of Young functions and ω a weight on G. Then, the weight $\omega_1^\sharp$, as constructed in the proposition with respect to the compact open subgroups $(K_i)_{i\in\mathbb{N}}$, is an L^Φ-weight.*

Proof. Indeed, by Proposition 3.28, $1/\omega_1^\sharp \in L^1(G) \cap L^\infty(G) \subseteq L^\Psi(G)$. It then follows from Proposition 3.19 that $L^\Phi(G, \omega_1^\sharp)$ is a Banach $*$-algebra and hence $\omega_1^\sharp$ is an L^Φ-weight. $\square$

We will now describe a method intrinsic to locally elliptic groups which allows one to determine if a weight is (equivalent to) a sub-additive weight.

Definition 3.30. Let G be a locally elliptic group and $(K_i)_{i\in\mathbb{N}}$ a sequence of compact open subgroups of G such that $K_i \leqslant K_{i+1}$ for each i and $G = \bigcup_{i=1}^{\infty} K_i$. Given a weight ω on G, define the ***variation*** of ω with respect to the $(K_i)_{i\in\mathbb{N}}$ to be the value

$$\mathrm{Var}(\omega, (K_i)_{i\in\mathbb{N}}) := \sup_{i\in\mathbb{N}}(\max_{x\in K_i} \omega(x) - \min_{y\in K_i} \omega(y)).$$

We say that ω has ***bounded variation*** if there exists such a sequence of compact open subgroups $(K_i)_{i\in\mathbb{N}}$ in G such that $\mathrm{Var}(\omega, (K_i)_{i\in\mathbb{N}}) < \infty$.

We then have the following result.

Proposition 3.31. *Let G be a locally elliptic group and ω a weight on G. If ω has bounded variation then it is equivalent to a sub-additive weight.*

Proof. Assume that G is locally elliptic and ω a weight on G with bounded variation. Let $(K_i)_{i\in\mathbb{N}}$ be a sequence of compact open subgroups of G such that $K_i \leqslant K_{i+1}$ for each i, $G = \bigcup_{i=1}^{\infty} K_i$, and $\mathrm{Var}(\omega, (K_i)_{i\in\mathbb{N}}) =: C < \infty$. Let $\omega^\sharp$ be the sub-additive weight constructed in Proposition 3.28 with respect to the compact open subgroups $(K_i)_{i\in\mathbb{N}}$. Then one checks easily that for all $x \in G$

$$\omega(x) \leqslant \omega^\sharp(x) \leqslant (C + 1)\omega(x)$$

which, by definition, means that ω is equivalent to the sub-additive weight $\omega^\sharp$. $\square$

In the following example, we show that a weight on a locally elliptic group need not have bounded variation nor be sub-additive.

Example 3.32. For $n \in \mathbb{N}$, let C_n denote the cyclic group of order n. Define the group $G := \bigoplus_{n=1}^{\infty} C_n$. Clearly G is a locally finite group and hence locally elliptic. We define a weight ω on G as follows: let x_n denote the canonical generator of the copy of C_n in G for each $n \in \mathbb{N}$. Then, given a general element $x = x_{n_1}^{m_1} \cdots x_{n_k}^{m_k} \in G$, where $1 \leqslant m_i \leqslant n_i - 1$ is an integer for each i, we define ω on x by

$$\omega(x) := (n_1)^{\max\{m_1, n_1 - m_1\}} \cdots (n_k)^{\max\{m_k, n_k - m_k\}}.$$

One checks that ω is a weight on G. The weight ω is not sub-additive since, for example,

$$\frac{\omega(x_{2n}^{n} x_{4n}^{2n})}{\omega(x_{2n}^{n}) + \omega(x_{4n}^{2n})} = \frac{(2n)^n (4n)^{2n}}{(2n)^n + (4n)^{2n}}$$

is unbounded as $n \to \infty$. This also implies that ω cannot have bounded variation by the previous proposition.

3.4. The Hermitian property

In this section we study the Hermitian and quasi-Hermitian properties for weighted Orlicz $*$-algebras on locally elliptic groups. We first start by proving Theorem 3.1 from the introduction.

Proof of Theorem 3.1. The proof is similar to the proof of Lemma 3.7 and Lemma 3.8 in [**82**]. Indeed, since $1/\omega \in L^{\Psi}(G)$ and hence $L^{\Phi}(G, \omega) \subseteq L^1(G)$, it follow by an identical argument to that of [**110**, Lemma 2.2] that there exists a constant $C_1 > 0$ such that

$$\|f\|_{L^1(G)} \leqslant C_1 \|f\|_{L^{\Phi}(G,\omega)}$$

for all $f \in L^{\Phi}(G, \omega)$. Then, given $f \in L^{\Phi}(G, \omega)$,

$$\nu_{L^1(G)}(f) = \lim_{n\to\infty} \|f^{*n}\|_{L^1(G)}^{1/n} \leqslant \lim_{n\to\infty} C_1^{1/n} \|f^{*n}\|_{L^{\Phi}(G,\omega)}^{1/n} = \nu_{L^{\Phi}(G,\omega)}(f).$$

Thus, we just need to show that $\nu_{L^{\Phi}(G,\omega)}(f) \leqslant \nu_{L^1(G)}(f)$ for all $f \in L^{\Phi}(G, \omega_1^{\sharp})$. We now fix $f \in L^{\Phi}(G, \omega_1^{\sharp})$. By Proposition 3.19(i), since $\omega_1^{\sharp}$ is sub-additive, there exists a constant $C_2 > 0$ such that

$$\|f * f\|_{L^{\Phi}(G,\omega_1^{\sharp})} \leqslant 2C_2 \|f\|_{L^{\Phi}(G,\omega_1^{\sharp})} \|f\|_{L^1(G)}.$$

By induction,

$$\left\|f^{*2n}\right\|_{L^{\Phi}(G,\omega_1^{\sharp})} \leqslant (2C_2)^n \|f\|_{L^{\Phi}(G,\omega_1^{\sharp})} \|f\|_{L^1(G)}^{2^n - 1}.$$

Then, we have that, since $\omega \leqslant \omega_1^\sharp$,

$$\begin{aligned}\nu_{L^\Phi(G,\omega)}(f) \leqslant \nu_{L^\Phi(G,\omega_1^\sharp)}(f) &= \lim_{n\to\infty} \left\| f^{*2^n} \right\|_{L^\Phi(G,\omega_1^\sharp)}^{2^{-n}} \\ &\leqslant \lim_{n\to\infty} (2C_2)^{n2^{-n}} \|f\|_{L^\Phi(G,\omega_1^\sharp)}^{2^{-n}} \|f\|_{L^1(G)}^{1-2^{-n}} \\ &= \|f\|_{L^1(G)} .\end{aligned}$$

It then follows that

$$\nu_{L^\Phi(G,\omega)}(f) = \lim_{n\to\infty} \nu_{L^\Phi(G,\omega)}(f^{*n})^{1/n} \leqslant \lim_{n\to\infty} \|f^{*n}\|_{L^1(G)}^{1/n} = \nu_{L^1(G)}(f).$$

This completes the proof. □

We now use Theorem 3.1 to prove Theorem 3.2 from the introduction. The proof of Theorem 3.2 relies heavily on [**49**, Lemma 3.1]. We state this lemma below for ease of reference, though we have slightly rephrased the statement for our purposes.

Lemma 3.33. [**49**, Lemma 3.1] *Let $\mathcal{A} \subseteq \mathcal{B}$ be a nested pair of Banach $*$-algebras that either both have a common identity element or both have no identity. Then the following are equivalent:*

(i) $\partial\sigma_{\mathcal{A}}(a) \subseteq \partial\sigma_{\mathcal{B}}(a)$ *for all* $a \in \mathcal{A}$*;*
(ii) $\partial\sigma_{\mathcal{A}}(a) \subseteq \sigma_{\mathcal{B}}(a)$ *for all* $a \in \mathcal{A}$*;*
(iii) $\nu_{\mathcal{A}}(a) = \nu_{\mathcal{B}}(a)$ *for all* $a \in \mathcal{A}$*.*

Furthermore, if $\mathcal{B}$ is Hermitian, then (*i*)*,* (*ii*) *and* (*iii*) *are equivalent to*

(iv) $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ *for all* $a \in \mathcal{A}$*.*

We now give the proof of Theorem 3.2.

Proof of Theorem 3.2. First we prove (*i*). By [**131**, Remark 4.10], $L^1(G)$ is quasi-Hermitian (i.e. $\sigma_{L^1(G)}(f) \subseteq \mathbb{R}$ for every self-adjoint $f \in C_c(G)$) since every compactly-generated subgroup of G is compact. Then, for a self-adjoint $f \in C_c(G) \subseteq L^\Phi(G,\omega)$, by Lemma 3.33 and Theorem 3.1, we have that $\partial\sigma_{L^\Phi(G,\omega)}(f) \subseteq \sigma_{L^1(G)}(f) \subseteq \mathbb{R}$. But this implies that $\sigma_{L^\Phi(G,\omega)}(f) \subseteq \mathbb{R}$ which completes the proof. Part (*ii*) is just a consequence of Proposition 3.19. □

Since every nilpotent group is Hermitian, it follows from Theorem 3.2 that every weighted Orlicz $*$-algebra on a nilpotent locally elliptic group with respect to a subadditive weight is Hermitian. It should be noted, however, that not every locally elliptic group is Hermitian. Indeed, there exists the following result.

THEOREM 3.34. [**71**] *There exists a class 2 solvable locally finite group Γ such that $\ell^1(\Gamma)$ is not Hermitian.*

We give the definition of two different non-Hermitian locally finite groups in Section 3.8 of this article, including the group from [**71**].

If Γ is the group in the theorem and A some non-discrete abelian locally elliptic group (e.g. $\mathbb{Q}_p$), then $A \times \Gamma$ is a non-discrete class 2 solvable locally elliptic group, and it is not Hermitian since Γ is a non-Hermitian quotient of $A \times \Gamma$. In particular, we have the following corollary.

Corollary 3.35. *There exist (infinitely many) non-discrete class 2 solvable locally elliptic groups which are not Hermitian.*

Any group of the form $A \times \Gamma$ has non-trivial torsion elements since Γ is locally-finite. Understanding what happens in the case of torsion-free solvable locally elliptic groups would be an interesting pursuit. We pose this as an open question.

Question 3.36. Does there exist a torsion-free class 2 solvable locally elliptic group which is not Hermitian?

Also, it would be interesting to have characterisations of Hermitianness for $L^1(G, \omega)$ which are independent of the Hermitian property of $L^1(G)$, when G is a locally elliptic group. Although we cannot say anything in general at the moment, we note the following result and an immediate corollary of it.

THEOREM 3.37. [**124**, Theorem 1] *Let G be a locally compact group and ω a subadditive weight on G. Suppose that there exists $p \in (0, \infty)$ such that $\omega^{-1} \in L^p(G)$. Then $L^1(G, \omega)$ is Hermitian.*

Corollary 3.38. *Let $G = \bigcup_{i=1}^{\infty} K_i$ be a locally elliptic group and let μ denote a left Haar measure on G. Suppose that $\mathbf{a} := (a_i)_{i=1}^{\infty} \subset \mathbb{R}_{\geqslant 0}$ is a non-decreasing sequence such that there exists $p \in (0, \infty)$ with*

$$\frac{\mu(K_1)}{a_1^p} + \sum_{i=2}^{\infty} \frac{\mu(K_i \backslash K_{i-1})}{a_i^p} < \infty.$$

Then $L^1(G, \omega_{\mathbf{a}})$ is Hermitian.

PROOF. Indeed, since

$$\omega_{\mathbf{a}}^{-1} = \frac{\chi_{K_1}}{a_1} + \sum_{i=2}^{\infty} \frac{\chi_{K_i \backslash K_{i-1}}}{a_i}$$

and

$$\begin{aligned}\|\omega_{\mathbf{a}}^{-1}\|_{L^p(G)}^p &= \int_G |\omega_{\mathbf{a}}^{-1}(x)|^p \, d\mu(x)\\ &= \int_{K_1} |\omega_{\mathbf{a}}^{-1}(x)|^p \, d\mu(x) + \sum_{i=2}^{\infty} \int_{K_i \backslash K_{i-1}} |\omega_{\mathbf{a}}^{-1}(x)|^p \, d\mu(x)\\ &= \frac{\mu(K_1)}{a_1^p} + \sum_{i=2}^{\infty} \frac{\mu(K_i \backslash K_{i-1})}{a_i^p},\end{aligned}$$

the result follows directly from these calculations and Theorem 3.37. □

We now note some other assumptions that imply Hermitianness for weighted Orlicz (or L^1) $*$-algebras on locally elliptic groups that do not require the weight to be sub-additive. First, we give the following definition.

Definition 3.39. Let G be a locally elliptic group and ω a weight on G. Suppose that there exist compact open subgroups $K_i \leqslant G$ ($i \in \mathbb{N} \cup \{0\}$) such that $K_i \leqslant K_{i+1}$ for each i, $G = \bigcup_{i=0}^{\infty} K_i$, and the weight

$$\omega'(n) := \max_{x \in K_{|n|}} \omega(x) \quad (n \in \mathbb{Z})$$

on $\mathbb{Z}$ satisfies the GRS condition. Then we say that ω satisfies the ***uniform GRS condition***.

We now prove the following lemma. The proof is basically identical to [**49**, Theorem 3.4] with some small modifications.

Lemma 3.40. *Let G be a locally elliptic group and ω a weight on G that satisfies the uniform GRS condition. Then, for all $f \in L^1(G,\omega)$, $\sigma_{L^1(G,\omega)}(f) = \sigma_{L^1(G)}(f)$.*

Proof. Let G be a locally elliptic group and ω a weight on G satisfying the uniform GRS condition. Let $K_i \leqslant G$ ($i \in \mathbb{N} \cup \{0\}$) be compact open subgroups of G such that $K_i \leqslant K_{i+1}$ for each i, $G = \bigcup_{i=0}^{\infty} K_i$ and the weight on $\mathbb{Z}$ defined by

$$\omega'(n) := \max_{x \in K_{|n|}} \omega(x) \quad (n \in \mathbb{Z})$$

has the GRS property. Then, for $f \in L^1(G,\omega)$, one computes by induction that

$$\begin{aligned}&\|f^{*n}\|_{L^1(G,\omega)} \leqslant \int_G \cdots \int_G |f(x_1)| \cdots |f(x_n)| \omega(x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n\\ &= \sum_{i_1,\ldots,i_n=0}^{\infty} \int_{K_{i_1} \backslash K_{i_1 - 1}} \cdots \int_{K_{i_n} \backslash K_{i_n - 1}} |f(x_1)| \cdots |f(x_n)| \omega(x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n\end{aligned}$$

where we use the convention that K_{-1} is the empty set.

Suppose that $x_j \in K_{i_j} \backslash K_{i_j - 1}$ for $j = 1, 2, \ldots, n$. Let i_k be the largest index of $i_1, \ldots, i_n$. Then, since $x_1 \cdots x_n \in K_{i_k}$,

$$\omega(x_1 \cdots x_n) \leqslant \max_{x \in K_{i_k}} \omega(x) = \omega'(i_k) \leqslant \omega'(i_1 + \cdots + i_n).$$

Then, let $a_i := \int_{K_i \backslash K_{i-1}} |f(x)|\, dx$ for each $i \in \mathbb{N} \cup \{0\}$ and $a := (a_n)_{n \in \mathbb{N} \cup \{0\}}$. Clearly $\|f\|_{L^1(G)} = \|a\|_{\ell^1(\mathbb{Z})}$. It then follows by the previous arguments that

$$\|f^{*n}\|_{L^1(G,\omega)} \leqslant \sum_{i_1, \ldots, i_n = 1}^{\infty} a_{i_1} \cdots a_{i_n} \omega(i_1 + \cdots + i_n) = \|a^{*n}\|_{\ell^1(\mathbb{Z}, \omega')}\,.$$

Hence, one computes that

$$\begin{aligned} \nu_{L^1(G,\omega)}(f) &= \lim_{n \to \infty} \|f^{*n}\|_{L^1(G,\omega)}^{1/n} \leqslant \lim_{n \to \infty} \|a^{*n}\|^{1/n} = \nu_{\ell^1(\mathbb{Z},\omega')}(a) \\ &= \nu_{\ell^1(\mathbb{Z})}(a) = \|a\|_{\ell^1(\mathbb{Z})} = \|f\|_{L^1(G)} \end{aligned}$$

where the third and fourth equalities follow from [**49**, Lemma 3.3]. Then

$$\nu_{L^1(G,\omega)}(f) = \lim_{n \to \infty} \nu_{L^1(G,\omega)}(f^n)^{1/n} \leqslant \lim_{n \to \infty} \|f^{*n}\|_{L^1(G)} = \nu_{L^1(G)}(f)$$

from which it follows that $\nu_{L^1(G,\omega)}(f) = \nu_{L^1(G)}(f)$. The result then follows by Lemma 3.33. ☐

We then have the following theorem.

Theorem 3.41. *Let G be a Hermitian locally elliptic group and (Φ, Ψ) a complementary pair of Young functions. The following hold:*

- *(i) If ω is an L^Φ-weight on G with $1/\omega \in L^\Psi(G)$ and ω has bounded variation, then $L^\Phi(G, \omega)$ is Hermitian;*
- *(ii) If ω is an arbitrary weight on G satisfying the uniform GRS condition, then $L^1(G, \omega)$ is Hermitian.*

Proof. Part *(i)* follows directly from Proposition 3.31 and Theorem 3.2. Part *(ii)* is a direct consequence of Lemma 3.40. ☐

3.5. Functional calculus and the Wiener property

In this section we prove Theorem 3.3. The proof requires us to develop a certain (smooth) functional calculus on a total part of a weighted Orlicz $*$-algebra on a locally elliptic group. Our arguments model those used in Section 4 and Section 5 of [**50**].

Throughout this section, unless otherwise specified, we assume the hypotheses of Theorem 3.3. In particular, G will be a locally elliptic group, (Φ, Ψ) a pair of complementary Young functions and ω an L^Φ-weight on G with $1/\omega \in L^\Psi(G)$.

Given $f \in L^\Phi(G,\omega)$, we define the function

$$u(f) := \sum_{k=1}^{\infty} \frac{f^{*k}}{k!}.$$

Since

$$\|u(f)\|_{L^\Phi(G,\omega)} \leqslant \sum_{k=1}^{\infty} \frac{\|f\|_{L^\Phi(G,\omega)}^k}{k!} = e^{\|f\|_{L^\Phi(G,\omega)}} - 1$$

it follows that $u(f) \in L^\Phi(G,\omega)$ for all $f \in L^\Phi(G,\omega)$.

The following result of Pytlik is critical to our results.

Lemma 3.42. [**124**, Lemma 4] *Let G be a locally compact group and ω a sub-additive weight on G. Suppose that there exists $p \in (0,\infty)$ such that $\omega^{-1} \in L^p(G)$. Then, for any self-adjoint $f \in L^1(G,\omega) \cap L^2(G)$ and $\gamma > \log_2((2p+2)/(p+2))$,*

$$\|u(inf)\|_{L^1(G,\omega)} = O(e^{n^\gamma}) \text{ as } n \to \infty.$$

We will now generalise this to the algebra $L^\Phi(G,\omega)$ on a locally elliptic group. To do this, we use the following result which is proved on page 142 of [**82**]. We include the proof due to its simplicity.

Lemma 3.43. [**82**, pg. 142] *For all $f \in L^\Phi(G,\omega)$ and $n \in \mathbb{N}$,*

$$u(nf) = nu(f) + \sum_{k=1}^{n-1} u(kf) * u(f).$$

PROOF. Note that, as a formal expression, $u(f) = e^f - 1$. Then,

$$\begin{aligned} u(nf) &= e^{nf} - 1 = e^{(n-1)f} * e^f - 1 = (u((n-1)f) + 1) * (u(f) + 1) - 1 \\ &= u((n-1)f) * u(f) + u((n-1)f) + u(f). \end{aligned}$$

A simple induction argument then gives the result. □

As a consequence, the following result holds.

Lemma 3.44. *Let G be a locally elliptic group, (Φ,Ψ) a complementary pair of Young functions and ω an L^Φ-weight on G. Let $(K_i)_{i\in\mathbb{N}}$ be a standard decomposition of G and $\omega_1^\sharp$ the weight as defined in Proposition 3.28 with respect to the compact open subgroups $(K_i)_{i\in\mathbb{N}}$. Then, for any self-adjoint $f \in L^\Phi(G,\omega) \cap L^1(G,\omega_1^\sharp) \cap L^2(G)$, and $\gamma > \log_2(4/3)$,*

$$\|u(inf)\|_{L^\Phi(G,\omega)} = O(e^{2n^\gamma}) \text{ as } n \to \infty.$$

PROOF. Throughout the proof, we fix a self-adjoint $f \in L^\Phi(G,\omega) \cap L^1(G,\omega_1^\sharp) \cap L^2(G)$ and $\gamma > \log_2(4/3)$. Since the weight $\omega_1^\sharp$ is a sub-additive weight on G and

$1/\omega_1^\sharp \in L^1(G)$, it follows by Lemma 3.42 that there exists a constant $C > 0$, such that for large n,

$$\|u(inf)\|_{L^1(G,\omega_1^\sharp)} \leqslant Ce^{n^\gamma}.$$

Then, by applying Lemma 3.43 and Proposition 3.21(iii), we have that

$$\begin{aligned}\|u(inf)\|_{L^\Phi(G,\omega)} &\leqslant n\,\|u(if)\|_{L^\Phi(G,\omega)} + \sum_{k=1}^{n-1} \|u(ikf)\|_{L^1(G,\omega)}\,\|u(if)\|_{L^\Phi(G,\omega)} \\ &\leqslant n\,\|u(if)\|_{L^\Phi(G,\omega)} + \sum_{k=1}^{n-1} \|u(ikf)\|_{L^1(G,\omega_1^\sharp)}\,\|u(if)\|_{L^\Phi(G,\omega)}.\end{aligned}$$

It follows that there exist constants C' and C'' such that

$$\begin{aligned}\|u(inf)\|_{L^\Phi(G,\omega)} &\leqslant C'(n + \sum_{k=1}^{n-1} \|u(ikf)\|_{L^1(G,\omega_1^\sharp)}) \\ &\leqslant C'(n + (n-1)\,\|u(i(n-1)f)\|_{L^1(G,\omega_1^\sharp)}) \\ &\leqslant C''ne^{n^\gamma} \leqslant C''e^{2n^\gamma}\end{aligned}$$

for n large. This is precisely what we needed to show. □

We are now able to construct our functional calculus. To do this, we define the following algebra of functions.

Definition 3.45. Given $0 < \gamma < 1$, define a weight on $\mathbb{Z}$ by $\omega_\gamma(n) := e^{2|n|^\gamma}$ $(n \in \mathbb{Z})$. Let $\tilde{A}_\gamma$ denote the algebra of 2π-periodic C^∞-functions $\varphi : \mathbb{R} \to \mathbb{R}$ with Fourier coefficients in $\ell^1(\mathbb{Z}, \omega_\gamma)$ under pointwise multiplication. In particular, the algebra $\tilde{A}_\gamma$ consists of functions of the form $\varphi(x) = \sum_{n\in\mathbb{Z}} \hat{\varphi}(n)e^{inx}$ such that $\sum_{n\in\mathbb{Z}} |\hat{\varphi}(n)|e^{2|n|^\gamma} < \infty$. Then, we let A_γ be the subalgebra of $\tilde{A}_\gamma$ consisting of those functions $\varphi \in \tilde{A}_\gamma$ with $\varphi(0) = 0$.

Before constructing our functional calculus, we will state the following fact about the algebra A_γ. This is a consequence of Lemma 1.24 and Theorem 2.11 in [**43**].

Proposition 3.46. [**43**] *For $0 < \gamma < 1$, the algebra A_γ contains functions of arbitrarily small support. Furthermore, for every $\epsilon > 0$ and every interval $[p, q] \subseteq (0, 2\pi)$ with $p + \epsilon < q - \epsilon$, there exists a function $\varphi \in A_\gamma$ satisfying the following properties:*

(i) $0 \leqslant \varphi \leqslant 1$*;*
(ii) $\operatorname{supp}(\varphi) \cap [0, 2\pi] \subseteq [p, q]$*;*
(iii) $\varphi(x) = 1$ *for all* $x \in [p + \epsilon, q - \epsilon]$.

We now prove the following proposition which gives the required functional calculus.

Proposition 3.47. *Let G be a locally elliptic group, (Φ, Ψ) a complementary pair of Young functions and ω an L^Φ-weight on G. Fix a γ with $\log_2(4/3) < \gamma < 1$. Then, for any $\varphi \in A_\gamma$ and self-adjoint $f \in L^\Phi(G,\omega) \cap L^1(G,\omega_1^\sharp) \cap L^2(G)$,*

$$\varphi\{f\} := \sum_{n\in\mathbb{Z}} \hat{\varphi}(n) u(inf)$$

has finite $L^\Phi(G,\omega)$-norm and hence converges to an element of $L^\Phi(G,\omega)$.

Proof. Indeed, by Lemma 3.44 and the definition of A_γ, we have that

$$\begin{aligned}\|\varphi\{f\}\|_{L^\Phi(G,\omega)} &\leqslant \sum_{n\in\mathbb{Z}} |\hat{\varphi}(n)|\, \|u(inf)\|_{L^\Phi(G,\omega)} \\ &\leqslant C \sum_{n\in\mathbb{Z}} |\hat{\varphi}(n)| e^{2|n|^\gamma} < \infty\end{aligned}$$

for some fixed constant $C > 0$. The result follows immediately. ☐

In particular, by the proposition, the algebra A_γ acts on compactly supported continuous self-adjoint functions in $L^\Phi(G,\omega)$.

Since A_γ is an algebra under pointwise multiplication, one can check, as done in [**50**, Section 4.7], that for $\varphi, \psi \in A_\gamma$ and a self-adjoint $f \in L^\Phi(G,\omega) \cap L^1(G,\omega_1^\sharp) \cap L^2(G)$,

$$(\varphi \cdot \psi)\{f\} = \varphi\{f\} * \psi\{f\}$$

and for any $*$-representation π of $L^\Phi(G,\omega)$,

$$\pi(\varphi\{f\}) = \varphi(\pi(f)),$$

where $\varphi(\pi(f))$ denotes the usual functional calculus of the bounded operator $\pi(f)$.

Now, we are going to work towards the proof of Theorem 3.3. First, we note that if G is locally elliptic and ω a weight on G, then, given a fixed compact neighbourhood K of the identity in G, there exists a bounded left approximate identity $(f_j)_{j\in J} \subseteq L^1(G,\omega)$ such that $f_j = f_j^*$ and $\operatorname{supp}(f_j) \subseteq K$ for each $j \in J$. Such an approximate identity exists by [**80**, Theorem 4.1].

We now prove the following result. The proof uses arguments given in [**50**, Section 5].

Lemma 3.48. *Let G be a locally elliptic group, ω a weight on G and $K \subseteq G$ a compact neighbourhood of the identity. Let $(f_j)_{j\in J} \subseteq L^1(G,\omega)$ be a bounded left approximate identity such that each f_j is self-adjoint and $\operatorname{supp}(f_j) \subseteq K$ for all j. Then, for any γ with $\log_2(4/3) < \gamma < 1$, there exists a function $\varphi \in A_\gamma$ such that, for all $g \in C_c(G)$, $\|\varphi\{f_j\} * g - g\|_{L^1(G,\omega)} \to 0$.*

Proof. Fix γ with $\log_2(4/3) < \gamma < 1$. By the arguments of [**50**, Section 5], it suffices to show that we can choose $\varphi \in A_\gamma$ with $\varphi(1) = 1$, so that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{|n|>N} |\hat{\varphi}(n)| \left\| e^{inf_j} * g \right\|_{L^1(G,\omega)} < \epsilon.$$

Now, let $(K_i)_{i\in\mathbb{N}}$ be a standard decomposition of G and $\omega_1^\sharp$ the weight as defined in Proposition 3.28 with respect to the compact open subgroups $(K_i)_{i\in\mathbb{N}}$. By Proposition 3.19(i), there exists a constant $C > 0$ such that

$$\begin{aligned}
&\left\| e^{inf_j} * g \right\|_{L^1(G,\omega)} \leqslant \left\| e^{inf_j} * g \right\|_{L^1(G,\omega_1^\sharp)} \\
&\leqslant \|g\|_{L^1(G,\omega_1^\sharp)} + \|u(inf_j) * g\|_{L^1(G,\omega_1^\sharp)} \\
&\leqslant \|g\|_{L^1(G,\omega_1^\sharp)} + C(\|u(inf_j)\|_{L^1(G,\omega_1^\sharp)} \|g\|_{L^1(G)} + \|g\|_{L^1(G,\omega_1^\sharp)} \|u(inf_j)\|_{L^1(G)}) \\
&\leqslant \|g\|_{L^1(G,\omega_1^\sharp)} + C(\|g\|_{L^1(G)} + \|g\|_{L^1(G,\omega_1^\sharp)}) \|u(inf_j)\|_{L^1(G,\omega_1^\sharp)} .
\end{aligned}$$

Since $\|u(inf_j)\|_{L^1(G,\omega_1^\sharp)} = O(e^{n^\gamma})$ as $n \to \infty$ by Lemma 3.42, and all other expressions in the last line are constants in the variable n, it follows that

$$\left\| e^{inf_j} * g \right\|_{L^1(G,\omega)} = O(e^{n^\gamma}) \text{ as } n \to \infty.$$

Then, by Proposition 3.46, we can choose a $\varphi \in A_\gamma$ such that

(i) $\varphi = 0$ in a neighbourhood of 0;
(ii) $\varphi(1) = 1$;
(iii) $\operatorname{supp}(\varphi) \cap [0, 2\pi]$ is compact.

For this φ we have that, for some constant $C > 0$,

$$\sum_{n\in\mathbb{Z}} |\hat{\varphi}(n)| \left\| e^{inf_j} * g \right\|_{L^1(G,\omega)} \leqslant C \sum_{n\in\mathbb{Z}} |\hat{\varphi}(n)| e^{|n|^\gamma} < \infty.$$

Thus, for any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that:

$$\sum_{|n|>N} |\hat{\varphi}(n)| \left\| e^{inf_j} * g \right\|_{L^1(G,\omega)} < \epsilon.$$

This φ then satisfies the lemma. ☐

A similar result can be proved for general weighted Orlicz $*$-algebras.

Lemma 3.49. *Let G be a locally elliptic group, (Φ, Ψ) a complementary pair of Young functions, ω an L^Φ-weight on G and $K \subseteq G$ a compact neighbourhood of the identity. Let $(f_j)_{j\in J} \subseteq L^1(G, \omega)$ be a bounded left approximate identity such that each f_j is self-adjoint and* $\operatorname{supp}(f_j) \subseteq K$ *for all j. Then, for any γ with* $\log_2(4/3) < \gamma < 1$, *there exists a function $\varphi \in A_\gamma$ such that, for all $f, g \in C_c(G)$,*

$$\|\varphi\{f_j\} * f * g - f * g\|_{L^\Phi(G,\omega)} \to 0.$$

PROOF. Fix a γ with $\log_2(4/3) < \gamma < 1$. By Lemma 3.48, we can find a function $\varphi \in A_\gamma$ such that $\|\varphi\{f_j\} * g - g\|_{L^1(G,\omega)} \to 0$ for all $g \in C_c(G)$. Then, let $f, g \in C_c(G)$. Since

$$\|\varphi\{f_j\} * f * g - f * g\|_{L^\Phi(G,\omega)} \leqslant \|\varphi\{f_j\} * f - f\|_{L^1(G,\omega)} \|g\|_{L^\Phi(G,\omega)}$$

by Proposition 3.19(i), and since

$$\|\varphi\{f_j\} * f - f\|_{L^1(G,\omega)} \to 0,$$

it follows that

$$\|\varphi\{f_j\} * f * g - f * g\|_{L^\Phi(G,\omega)} \to 0. \qquad \square$$

We now prove Theorem 3.3. The argument already exists in literature (see [**86**, Section 3, (10)] and [**82**, Theorem 6.3]), but we provide it here for completeness.

Proof of Theorem 3.3.

For the proof we fix $\log_2(4/3) < \gamma < 1$. Let $I \subseteq L^\Phi(G,\omega)$ be a proper non-trivial closed two-sided ideal. Let $(f_j)_{j\in J} \subseteq L^1(G,\omega)$ and $\varphi \in A_\gamma$ be defined as in Lemma 3.49.

Now, we claim that $\varphi\{f_j\} \notin I$ for at least one j. Indeed, suppose the contrary. Then, $\varphi\{f_j\} * f * g \in I$ for all $j \in J$ and $f, g \in C_c(G)$ since I is an ideal. By Lemma 3.49 and the fact that I is closed, it follows that $f * g \in I$ for all $f, g \in C_c(G)$. This implies that $I = L^\Phi(G,\omega)$ by density of $C_c(G)$. This contradicts the fact that I is assumed to be a proper ideal.

Thus, we may now fix $j \in J$ such that $\varphi\{f_j\} \notin I$. Let $\psi \in A_\gamma$ be a function which is identically 1 on the support of φ, which exists by Proposition 3.46. Then, $\psi\{f_j\} * \varphi\{f_j\} = (\psi\varphi)\{f_j\} = \varphi\{f_j\}$. In particular, the element $\psi\{f_j\} * \varphi\{f_j\} = \varphi\{f_j\}$ is a non-trivial element of the quotient $L^\Phi(G,\omega)/I$, and it follows that the image of $\psi(f_j)$ in the quotient $L^\Phi(G,\omega)/I$ contains 1 in its spectrum. Thus, $L^\Phi(G,\omega)/I$ is not a radical Banach algebra, and hence there exists a non-trivial algebraically irreducible representation of $L^\Phi(G,\omega)/I$ [**114**, Theorem 2.3.3]. Lifting this representation to a representation of $L^\Phi(G,\omega)$ then gives a non-trivial algebraically irreducible representation of $L^\Phi(G,\omega)$ that annihilates I. This completes the proof. $\square$

3.6. The $*$-regular property and representations of locally elliptic groups

Let G be a locally elliptic group, (Φ, Ψ) a complementary pair of Young functions with $\Phi \in \Delta_2$, and ω an L^Φ-weight such that $1/\omega \in L^\Psi(G)$. As mentioned in the introduction, since $\Phi \in \Delta_2$, the (vector space) dual of $L^\Phi(G,\omega)$ is $L^\Phi(G,\omega)^* = L^\Psi(G,\omega^{-1})$ and it follows that every linear functional $\lambda \in L^\Phi(G,\omega)^*$ is of the form

$$\lambda(f) = \int_G f(x)g(x)\,dx \quad (f \in L^\Phi(G,\omega))$$

for some $g \in L^\Psi(G,\omega^{-1})$. Then, one checks that the arguments in [**82**, Section 2.4] work identically for the algebra $L^\Phi(G,\omega)$, in particular, the map

$$\pi \mapsto \left(f \mapsto \int_G f(x)\pi(x)\,dx\right)$$

provides a bijection between unitary representations of G and unitary representations (i.e. non-degenerate $*$-representations) of $L^\Phi(G,\omega)$. Furthermore, this map preserves irreducibility and equivalence of representations. As a consequence of this, the space $\widehat{L^\Phi(G,\omega)}$ can be identified with $\widehat{G}$. This also implies that $C^*(L^\Phi(G,\omega)) = C^*(G)$ since $C_c(G) \subseteq L^\Phi(G,\omega) \subseteq L^1(G)$; this is precisely Theorem 3.5(i).

In the following, given a unitary representation π of G and f an element of $L^\Phi(G,\omega)$ (resp. $L^1(G)$), we will use the convention that π also denotes the $*$-representation of $L^\Phi(G,\omega)$ (resp. $L^1(G)$) defined on f by

$$\pi(f) := \int_G f(x)\pi(x)\,dx.$$

Similarly, π will also be used to denote the corresponding unitary representation of the group C^*-algebra $C^*(G)$.

We now prove the following result. The statement and proof model that of [**45**, Proposition 5.2].

Proposition 3.50. *Let G be a locally elliptic group, (Φ,Ψ) a complementary pair of Young functions with $\Phi \in \Delta_2$, and ω an L^Φ-weight on G with $1/\omega \in L^\Psi(G)$. Let $C \subseteq \widehat{G}$ and fix $\rho \in \widehat{G}$. The following are equivalent:*

(i) $\cap_{\pi \in C} \ker_{C^*(G)}(\pi) \subseteq \ker_{C^*(G)}(\rho)$;
(ii) $\cap_{\pi \in C} \ker_{L^1(G)}(\pi) \subseteq \ker_{L^1(G)}(\rho)$;
(iii) $\cap_{\pi \in C} \ker_{L^\Phi(G,\omega)}(\pi) \subseteq \ker_{L^\Phi(G,\omega)}(\rho)$;
(iv) $\|\rho(f)\|_{\mathcal{B}(\mathcal{H}_\rho)} \leqslant \sup_{\pi \in C} \|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\pi)}$ *for all* $f \in C_c(G)$;
(v) $\|\rho(f)\|_{\mathcal{B}(\mathcal{H}_\rho)} \leqslant \sup_{\pi \in C} \|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\pi)}$ *for all* $f \in L^\Phi(G,\omega)$;
(vi) $\|\rho(f)\|_{\mathcal{B}(\mathcal{H}_\rho)} \leqslant \sup_{\pi \in C} \|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\pi)}$ *for all* $f \in L^1(G)$.

PROOF. Since $1/\omega \in L^\Psi(G)$, we have that $L^\Phi(G,\omega) \subseteq L^1(G) \subseteq C^*(G)$, and this directly implies that $(i) \implies (ii) \implies (iii)$.

For every group of polynomial growth, and in particular, every locally elliptic group G, $L^1(G)$ is $*$-regular [**13**, Satz 2] i.e. $\mathrm{Prim}_*(L^1(G))$ is homeomorphic to $\mathrm{Prim}_*(C^*(G))$. Consequently, (ii) implies (i).

Since $C_c(G) \subseteq L^\Phi(G,\omega) \subseteq L^1(G)$, we clearly have that $(vi) \implies (v) \implies (iv)$. On the other hand, since $C_c(G)$ is dense in $L^1(G)$ and $\|\sigma(f)\|_{\mathcal{B}(\mathcal{H}_\sigma)} \leqslant \|f\|_{L^1(G)}$ for all $\sigma \in \widehat{G}$ and $f \in L^1(G)$, it follows immediately that $(iv) \implies (vi)$. Thus (iv), (v) and (vi) are equivalent.

Clearly (vi) implies (ii) and hence (i) by $*$-regularity of $L^1(G)$.

Thus, to complete the proof, we just need to show that $(iii) \implies (iv)$. We follow a similar argument to that used in the proof of [**45**, Proposition 5.2]. Assume, for a contradiction, that (iv) is false i.e. that there exists $g \in C_c(G)$ such that

$$\sup_{\pi \in C} \|\pi(g)\|_{\mathcal{B}(\mathcal{H}_\pi)} < \|\rho(g)\|_{\mathcal{B}(\mathcal{H}_\rho)}\,.$$

We may assume that g is self-adjoint and $\|g\|_{L^1(G)} \leqslant 1$. Let γ be such that $\log_2(4/3) < \gamma < 1$. Recall from Section 3.5 that the algebra A_γ acts by functional calculus on the self-adjoint compactly supported continuous functions in $L^\Phi(G,\omega)$, and in particular, on the function g. Now choose $\varphi \in A_\gamma$ that is zero on a neighbourhood of the interval

$$[-\sup_{\pi \in C} \|\pi(g)\|_{\mathcal{B}(\mathcal{H}_\pi)}\,, \sup_{\pi \in C} \|\pi(g)\|_{\mathcal{B}(\mathcal{H}_\pi)}] \subseteq [-1,1]$$

and such that $\varphi(\|\rho(g)\|_{\mathcal{B}(\mathcal{H}_\rho)}) = 1$. Such a function exists by Proposition 3.46. Then, it follows that φ is zero on the spectrum of $\pi(g)$ for every $\pi \in C$, hence, $\pi(\varphi\{g\}) = \varphi(\pi(g)) = 0$ for all $\pi \in C$. Also, since $\|\rho(g)\|_{\mathcal{B}(\mathcal{H}_\rho)}$ is in the spectrum of $\rho(g)$ and $\varphi(\|\rho(g)\|_{\mathcal{B}(\mathcal{H}_\rho)}) = 1$, $\rho(\varphi\{g\}) = \varphi(\rho(g)) \neq 0$. Thus $\varphi\{g\} \notin \ker_{L^\Phi(G,\omega)}(\rho)$ but $\varphi\{g\} \in \cap_{\pi \in C} \ker_{L^\Phi(G,\omega)}(\pi)$. This contradicts (iii) and hence the result follows. $\square$

The proof of Theorem 3.5(ii) is a direct consequence of the proposition and the definition of the hull-kernel topology on each of the respective primitive ideal spaces.

In the following proposition, if $\mathcal{A}$ is a Banach $*$-algebra, then $\mathrm{Max}(\mathcal{A})$ will denote the space of all maximal closed two-sided ideals of $\mathcal{A}$ equipped with the hull-kernel topology.

Proposition 3.51. *Let G be a locally elliptic group, (Φ,Ψ) a complementary pair of Young functions with $\Phi \in \Delta_2$, and ω an L^Φ-weight on G with $1/\omega \in L^\Psi(G)$. Suppose that $L^\Phi(G,\omega)$ is Hermitian. Then, the following are equivalent:*

- *(i)* $\mathrm{Prim}(C^*(G)) = \mathrm{Prim}_*(C^*(G))$ *is* T_1*;*
- *(ii)* $\mathrm{Prim}_*(L^1(G))$ *is* T_1*;*
- *(iii)* $\mathrm{Prim}_*(L^\Phi(G,\omega))$ *is* T_1*;*
- *(iv)* $\mathrm{Prim}_*(L^\Phi(G,\omega)) \subseteq \mathrm{Max}(L^\Phi(G,\omega))$*;*
- *(v)* $\mathrm{Prim}_*(L^\Phi(G,\omega)) = \mathrm{Max}(L^\Phi(G,\omega))$*.*

Proof. The equivalence of (i), (ii) and (iii) follows directly from Theorem 3.5. Clearly (v) implies (iv). Since $L^\Phi(G,\omega)$ is Hermitian by assumption, it is Wiener by Theorem 3.3, which implies that $\mathrm{Max}(L^\Phi(G,\omega)) \subseteq \mathrm{Prim}_*(L^\Phi(G,\omega))$. Thus (iv) implies (v).

Finally, we need to prove the equivalence of (iii) and (iv). First, let's suppose that (iii) is true and prove (iv). Suppose for a contradiction that there is an ideal J in $\mathrm{Prim}_*(L^\Phi(G,\omega))$ which is not maximal amongst closed two-sided ideals. Then, there exists a proper closed (two-sided) ideal $J' \subseteq L^\Phi(G,\omega)$ properly containing J. Since $L^\Phi(G,\omega)$ is Hermitian and hence Wiener by Theorem 3.3, $\mathrm{hull}_*(\{J'\})$ is non-empty and hence contains an ideal I in $\mathrm{Prim}_*(L^\Phi(G,\omega))$. In particular, $\mathrm{hull}_*(\{J\}) \supseteq \{I, J\}$, from which it follows, by definition of the hull-kernel topology, that J is not a closed point of $\mathrm{Prim}_*(L^\Phi(G,\omega))$. This contradicts the fact that $\mathrm{Prim}_*(L^\Phi(G,\omega))$ is T_1.

We now suppose that (iv) is true and prove (iii). We need to show that under the assumption of (iv), every point in $\mathrm{Prim}_*(L^\Phi(G,\omega))$ is closed. Let $J \in \mathrm{Prim}_*(L^\Phi(G,\omega))$. Then, since J is maximal by the assumption of (iv), it follows that $\mathrm{hull}_*(\ker(\{J\})) = \{J\}$, which implies that J is a closed point in $\mathrm{Prim}_*(L^\Phi(G,\omega))$ by definition of the hull-kernel topology. $\square$

It is an ongoing area of research to determine for which locally elliptic groups G does the topology on $\mathrm{Prim}(C^*(G))$ satisfy the T_1 separation axiom. We note that if G is either 2-step-nilpotent, or nilpotent and has a compact open normal subgroup, then it is already known that $\mathrm{Prim}(C^*(G))$ is T_1 [**120, 26**].

3.7. Minimal ideals of a given hull

Let G be a locally elliptic group, (Φ,Ψ) a complementary pair of Young functions with $\Phi \in \Delta_2$, and ω an L^Φ-weight such that $1/\omega \in L^\Psi(G)$. Throughout this section we fix γ with $\log_2(4/3) < \gamma < 1$ so that we have a functional calculus of A_γ acting on compactly supported self-adjoint functions in $L^\Phi(G,\omega)$. Now, given $C \subseteq \mathrm{Prim}_*(L^\Phi(G,\omega))$ closed, we define the following set, which we view as a subset of $L^\Phi(G,\omega)$:

$$\begin{aligned} m(C) := \{\varphi\{f\} : f = f^* \in C_c(G), \|f\|_{L^1(G)} \leqslant 1, \varphi \in A_\gamma, \varphi = 0 \text{ on a} \\ \text{neighbourhood of } [-\sup_{\ker(\pi)\in C} \|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\pi)}, \sup_{\ker(\pi)\in C} \|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\pi)}]\}. \end{aligned}$$

Let $j(C)$ denote the closed two-sided ideal of $L^\Phi(G,\omega)$ generated by the set $m(C)$. Then, a similar argument to Lemma 1 and Theorem 1 in [**96**] give the following theorem. We repeat the proof for completeness and to clarify the small changes that need to be made to the argument.

THEOREM 3.52. *Let G be a locally elliptic group, (Φ, Ψ) a complementary pair of Young functions with $\Phi \in \Delta_2$, and ω an L^Φ-weight such that $1/\omega \in L^\Psi(G)$. Suppose that $L^\Phi(G,\omega)$ is Hermitian. Then, for every closed set $C \subseteq \mathrm{Prim}_*(L^\Phi(G,\omega))$, the closed two-sided ideal $j(C) \subseteq L^\Phi(G,\omega)$ satisfies the following properties:*

(i) $\mathrm{hull}_*(j(C)) = C$;

(ii) For every closed two-sided ideal $I \subseteq L^\Phi(G,\omega)$ with $\mathrm{hull}_*(I) \subseteq C$, $j(C) \subseteq I$.

PROOF. (*i*) Let $\ker_{L^\Phi(G,\omega)}(\pi) \in C$ and $\varphi\{f\} \in m(C)$. Then, since $\pi(\varphi\{f\}) = \varphi(\pi(f))$, and φ is zero on the spectrum of $\pi(f)$, it follows that $\pi(\varphi\{f\}) = 0$. Thus, $m(C) \subseteq \ker_{L^\Phi(G,\omega)}(\pi)$ for all $\ker_{L^\Phi(G,\omega)}(\pi) \in C$, and it follows that $\mathrm{hull}_*(m(C)) \subseteq C$, which implies that $\mathrm{hull}_*(j(C)) \subseteq C$.

Conversely, suppose that $\ker_{L^\Phi(G,\omega)}(\rho) \notin C$ for some $\rho \in \widehat{L^\Phi(G,\omega)}$. Then, by Proposition 3.50, there exists $f \in C_c(G)$ self-adjoint with $\|f\|_{L^1(G)} < 1$ such that

$$\sup_{\ker(\pi)\in C} \|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\pi)} < \|\rho(f)\|_{\mathcal{B}(\mathcal{H}_\rho)}.$$

Choose $\varphi \in A_\gamma$ such that φ is zero in a neighbourhood of the interval

$$[-\sup_{\ker(\pi)\in C} \|\pi(f)\|_{B(\mathcal{H}_\pi)}, \sup_{\ker(\pi)\in C} \|\pi(f)\|_{B(\mathcal{H}_\pi)}]$$

and $\varphi(\|\rho(f)\|_{\mathcal{B}(\mathcal{H}_\rho)}) = 1$. It then follows that $\rho(\varphi\{f\}) = \varphi(\rho(f)) \neq 0$ and hence $\ker(\rho)$ does not contain $m(C)$. Thus it follows that $\mathrm{hull}_*(j(C)) = C$.

(*ii*) Let $I \subseteq L^\Phi(G,\omega)$ be a closed two-sided ideal with $\mathrm{hull}_*(I) \subseteq C$. Let $\varphi\{f\} \in m(C)$. Choose non-trivial $\psi \in A_\gamma$ which is identically 1 on the support of φ, which exists by Proposition 3.46. Then, $\psi\{f\} * \varphi\{f\} = \varphi\{f\}$ since $\psi\varphi = \varphi$. Also, it is clear that $\psi\{f\} \in m(C)$ and $\mathrm{hull}_*(\{\psi\{f\}\}) \supseteq \mathrm{hull}_*(m(C)) = C$. Then [**96**, Lemma 2] gives the result. ☐

3.8. Examples and open questions

3.8.1. Locally elliptic groups with the bounded index property. We start with the following definition of a locally elliptic group having the bounded index property.

Definition 3.53. Let G be a locally elliptic group. The group G has the ***bounded index property*** if there exist compact open subgroups $K_i \leqslant G$ for each $i \in \mathbb{N}$ such that $K_i \leqslant K_{i+1}$ for each i, $G = \bigcup_{i=1}^\infty K_i$ and $\sup_{n\in\mathbb{N}}[K_{n+1} : K_n] < \infty$.

We will give a number of examples of locally elliptic groups with the bounded index property throughout this section. First, we would like to prove the following result, which gives a canonical way of constructing L^p-weights, or more generally, L^Φ-weights on these groups for some Young function Φ.

Proposition 3.54. *Let* $G = \bigcup_{i=1}^{\infty} K_i$ *be a locally elliptic group and suppose that* $M := \sup_{n\in\mathbb{N}}[K_{n+1} : K_n] < \infty$*. Given* $f \in \ell^1(\mathbb{N})$ *and* $q \in [1,\infty)$*, define the sub-additive weight*

$$\omega_{f,q} := \chi_{K_1} + \sum_{n=1}^{\infty} \left(\frac{M^n}{f(n)}\right)^{1/q} \chi_{K_{n+1}\backslash K_n}.$$

Then, $1/\omega_{f,q} \in L^q(G)$*. In particular, we have the following:*

(i) If $q = 1$*, then* $\omega_{f,1}$ *is an* L^{Φ}*-weight on* G *for any Young function* Φ*;*
(ii) If $q > 1$ *and* $p \in (1,\infty)$ *such that* $1/p + 1/q = 1$*, then* $\omega_{f,q}$ *is an* L^p*-weight.*

Proof. The inverse of $\omega_{f,q}$ is given by

$$1/\omega_{f,q} = \chi_{K_1} + \sum_{n=1}^{\infty} \left(\frac{f(n)}{M^n}\right)^{1/q} \chi_{K_{n+1}\backslash K_n}.$$

We now show that $1/\omega_{f,q} \in L^q(G)$. Let μ be the Haar measure on G. We may assume that μ has been normalised so that $\mu(K_1) = 1$. Then,

$$\mu(K_2\backslash K_1) = [K_2 : K_1]\mu(K_1) - \mu(K_1) < M\mu(K_1) = M,$$

and by induction, for $n > 2$,

$$\mu(K_{n+1}\backslash K_n) = [K_{n+1} : K_n]\mu(K_n) - \mu(K_n) < M\mu(K_n) \leqslant M^n.$$

One then computes

$$\begin{aligned}\int_G 1/\omega_{f,q}^q(x)\,dx &= \int_{K_1} 1/\omega_{f,q}^q(x)\,dx + \sum_{n=1}^{\infty} \int_{K_{n+1}\backslash K_n} 1/\omega_{f,q}^q(x)\,dx \\ &< 1 + \sum_{n=1}^{\infty} f(n) < \infty.\end{aligned}$$

Thus, $1/\omega_{f,q} \in L^q(G)$. If $q > 1$, it follows by Lemma 3.27 that $\omega_{f,q}$ is an L^p-weight on G since it is sub-additive, which is *(ii)*. On the other hand, if $q = 1$ and (Φ, Ψ) a complementary pair of Young functions, then $1/\omega_{f,1} \in L^1(G) \cap L^{\infty}(G) \subseteq L^{\Psi}(G)$, and since $\omega_{f,1}$ is sub-additive, it follows by Proposition 3.19 that $L^{\Phi}(G, \omega_{f,q})$ is a Banach $*$-algebra. Thus $\omega_{f,1}$ is an L^{Φ}-weight for any Young function Φ. This is *(i)*. ☐

3.8.2. Contraction groups. We begin with the definition of a contraction group.

Definition 3.55. Let G be a locally compact group and α a bi-continuous automorphism of G. The pair (G, α) is called a ***contraction group*** if for all $x \in G$, $\alpha^n(x) \to \mathrm{id}_G$ as $n \to \infty$.

We will now give some examples of contraction groups.

Example 3.56. (i) The additive group $(\mathbb{Q}_p, +)$ with automorphism multiplication by p. This example can be extended to any local field.

(ii) Let F be a finite group. The group $(\bigoplus_{\mathbb{Z}_{<0}} F) \times (\prod_{\mathbb{Z}_{\geqslant 0}} F)$ equipped with the "right-shift" automorphism is a torsion contraction group.

(iii) The group $U_n(\mathbb{Q}_p)$ of n-dimensional unipotent matrices over $\mathbb{Q}_p$ equipped with the automorphism which is conjugation by the diagonal matrix

$$\mathrm{diag}(1, p, p^2, \ldots, p^{n-1}).$$

The field $\mathbb{Q}_p$ can also be replaced with any local field in this example, provided you change the automorphism appropriately.

Contraction groups have a fundamental importance in the structure theory of totally disconnected locally compact groups [**6**] and their structure theory has been studied extensively in the papers [**58, 59, 60**]. The following structure theorem about contraction groups is key to their study.

Theorem 3.57. [**58, 59**] *Let (G, α) be a contraction group. Then, there exists distinct primes $p_1, \ldots, p_n$ and unipotent linear algebraic groups $G_{p_1}, \ldots, G_{p_n}$ over $\mathbb{Q}_{p_1}, \ldots, \mathbb{Q}_{p_n}$ respectively such that*

$$G \cong G_0 \times G_{p_1} \times \cdots \times G_{p_n} \times \mathsf{tor}(G)$$

where G_0 denotes the connected component of the identity in G and $\mathsf{tor}(G)$ the subgroup of torsion elements. Furthermore, G_0 is a connected simply-connected nilpotent real Lie group.

If we further assume that the contraction group G is totally disconnected, then we may choose $U \leqslant G$ a compact open subgroup, and it follows from α being contractive that $G = \bigcup_{n=1}^{\infty} \alpha^{-n}(U)$. Thus, G is locally elliptic since $\alpha^n(U)$ is compact for each n, and G has the bounded index property since the indices $[\alpha^{-n-1}(U) : \alpha^{-n}(U)]$ are constant with n varying. We state this below.

Proposition 3.58. *Every totally disconnected locally compact contraction group is locally elliptic and has the bounded index property.*

By Theorem 3.57, every torsion-free contraction group is nilpotent and hence it is Hermitian by [**95**]. However, in contrast, torsion contraction groups need not be nilpotent or even solvable. For example, if F is a non-solvable finite group, then the group

$$G := \left(\bigoplus_{\mathbb{Z}_{<0}} F\right) \times \left(\prod_{\mathbb{Z}_{\geqslant 0}} F\right)$$

is a non-solvable torsion contraction group when equipped with the right-shift automorphism as its contractive automorphism. This group, however, is also Hermitian since it is an $[FC]^-$-group [**91**, Theorem 2.6]. But torsion contraction groups need not be $[FC]^-$-groups either, with the Heisenberg group over $\mathbb{F}_p(\!(t)\!)$ providing an example of a torsion contraction group that is not an $[FC]^-$-group.

We thus pose the following question, which is an interesting question in the context of the harmonic analysis of totally disconnected locally compact groups.

Question 3.59. Is every totally disconnected locally compact contraction group Hermitian and hence Wiener?

Furthermore, it is a non-trivial question as to whether the direct product of two Hermitian groups is Hermitian (*c.f.* [**14**]). Thus, we also pose the following question which one would need to understand to answer Question 3.59.

Question 3.60. If (G, α) is a contraction group and $\mathsf{tor}(G)$ is Hermitian/Wiener, is G also Hermitian/Wiener?

The author is pursuing a solution to these questions on contraction groups in ongoing work, and this work ties in with recent research on the representation theory of contraction groups [**27, 22**].

3.8.3. Unipotent linear algebraic group over non-archimedean local fields.

Let k be a non-archimedean local field and define $U_n(k)$ to be the group of all upper-triangular n-dimensional unipotent matrices over k. Given $x \in k$ with absolute value < 1, one checks that conjugation by the diagonal matrix $\operatorname{diag}(1, x, x^2, \ldots, x^{n-1})$ is a contractive automorphism of $U_n(k)$. Thus, it follows that $U_n(k)$ is a locally elliptic group with the bounded index property by the discussion in the previous section, and since these properties pass to closed subgroups, we get the following result.

Proposition 3.61. *Let N be a group of n-dimensional unipotent matrices over a non-archimedean field. The group N is locally-elliptic and has the bounded index property.*

Now let's assume that N is a group of n-dimensional unipotent matrices over a p-adic field for some prime p. Let $\mathfrak{n}$ denote the Lie algebra of N and $\mathfrak{n}^*$ the vector space dual of $\mathfrak{n}$. The group N acts on $\mathfrak{n}^*$ by the so called ***codajoint action***

$$(\mathrm{Ad}^*(g)\lambda)(X) := \lambda(\mathrm{Ad}(g^{-1})X) \quad (g \in N, X \in \mathfrak{n}, \lambda \in \mathfrak{n}^*)$$

where Ad denotes the usual adjoint action of N on $\mathfrak{n}$.

The following result is well known.

THEOREM 3.62. [**105, 19**] *Let N be a group of n-dimensional unipotent matrices over a p-adic field. Then the following hold:*

(i) N is CCR;
(ii) The spaces $\widehat{N}$, $\mathrm{Prim}(C^*(N))$ and $\mathfrak{n}^*/\mathrm{Ad}^*(N)$ are homeomorphic.

In particular, we have the following corollary by the results in this article.

Corollary 3.63. *Let N be a group of n-dimensional unipotent matrices over a p-adic field. Let (Φ, Ψ) be a complementary pair of Young functions with $\Phi \in \Delta_2$ and ω an L^Φ-weight on N with $1/\omega \in L^\Psi(N)$. Then, there exists a homeomorphism*

$$\eta : \mathrm{Prim}_*(L^\Phi(N,\omega)) \to \mathfrak{n}^*/\mathrm{Ad}^*(N).$$

Maintaining the notation as in the corollary, let's suppose that $L^\Phi(N,\omega)$ is Hermitian (which is the case if ω is sub-additive, for example) and suppose that I is a closed ideal of $L^\Phi(N,\omega)$ with $\mathrm{hull}_*(I) := C \subseteq \mathrm{Prim}_*(L^\Phi(N,\omega))$. Let $j(C)$ be the ideal of $L^\Phi(N,\omega)$ as defined in Theorem 3.52. Then, the ideal I must sit between $\ker(C)$ and $j(C)$ i.e. $j(C) \subseteq I \subseteq \ker(C)$. In particular, understanding the closed ideals of $L^\Phi(G,\omega)$ with hull equal to C would require one to understand the ideal theory of the algebra $\ker(C)/j(C)$.

The set $C \subseteq \mathrm{Prim}_*(L^\Phi(N,\omega))$ is called a ***set of synthesis*** if $\ker(C)/j(C)$ is trivial, or equivalently, $\ker(C)$ is the only closed ideal of $L^\Phi(N,\omega)$ with hull C. The property of $L^\Phi(N,\omega)$ being Wiener is equivalent to the empty set being a set of synthesis, so under our assumption that $L^\Phi(N,\omega)$ is Hermitian, it follows that the empty set is a set of synthesis by Theorem 3.3. Also, by the previous corollary, we may identify C with a subset of $\mathfrak{n}^*/\mathrm{Ad}^*(N)$. We thus pose the following problem.

Question 3.64. Let N be a group of n-dimensional unipotent matrices over a p-adic field. Let (Φ, Ψ) be a complementary pair of Young functions and ω an L^Φ-weight on N with $1/\omega \in L^\Psi(N)$ and $L^\Phi(N,\omega)$ Hermitian. Given

$$C \subseteq \mathfrak{n}^*/\mathrm{Ad}^*(N) \cong \mathrm{Prim}_*(L^\Phi(N,\omega)),$$

can geometric properties of C, when viewed as a subset of $\mathfrak{n}^*/\mathrm{Ad}^*(N)$, be linked to algebraic properties of the algebra $\ker(C)/j(C)$? In particular, what geometric properties of C imply that it is a set of synthesis?

We will now note some results that are known in the case of connected nilpotent Lie groups, in which this question is motivated by. We now suppose that N_0 is a connected nilpotent (real) Lie group. The entire discussion of this section regarding Orlicz $*$-algebras on unipotent p-adic groups goes through when the (unipotent p-adic) group is replaced with the connected nilpotent Lie group N_0 and ω a sub-exponential weight on N_0 [**111**]. In particular, $L^1(N_0)$ contains a minimal ideal $j(C)$ for each hull $C \subseteq \mathrm{Prim}_*(L^1(N_0))$.

It is known that for every one-point set $\{J\} \subseteq \mathrm{Prim}_*(L^1(N_0))$, the algebra $\ker(\{J\})/j(\{J\}) = J/j(\{J\})$ is nilpotent [**97**]. Furthermore, if N_0 is 2-step-nilpotent, then all singleton sets are sets of synthesis, but this does not hold if N_0 is 3-step-nilpotent [**98**]. In particular, in some sense, if N_0 is 2-step-nilpotent, then closed ideals of $L^1(N_0)$ whose hulls are singleton sets are classified. More generally, if N_0 is not 2-step-nilpotent, then the set $\{J\}$ is a set of synthesis provided that $\{J\} \subseteq \mathrm{Prim}_*(L^1(N_0))$ corresponds to a "flat orbit" in $\mathfrak{n}_0^*/\mathrm{Ad}^*(N_0)$. This was more recently generalised to weighted L^1-algebras on connected nilpotent Lie groups in [**94**].

It would be good to try and replicate these results for weighted Orlicz ∗-algebras on unipotent p-adic groups, or more generally, torsion-free contraction groups. Even treating the case of the full group algebra $L^1(N)$ for a unipotent p-adic group N would be novel.

3.8.4. Non-Hermitian locally finite groups.

Here we will write down the definitions of two non-Hermitian locally finite groups found in the papers by Fountain-Ramsey-Williamson [**53**] and Hulanicki [**71**]. Expanding on these two examples, it would be an interesting question for future research to characterise the Hermitian locally finite groups.

3.8.4.1. *The Fountain-Ramsay-Williamson group.* The Fountain-Ramsay-Williamson group is the locally finite discrete group G with generators $\{x_n : n \in \mathbb{N}\}$ subject to the following relations:

(i) $x_n^2 = \mathrm{id}_G$ for all $n \in \mathbb{N}$;
(ii) $x_l x_m x_n x_m = x_m x_n x_m x_l$ for all $l, m < n \in \mathbb{N}$.

It is a consequence of [**53**, Section 5 & 6] that the group G is locally finite and not Hermitian.

3.8.4.2. *The Leptin-Hulanicki group.* Let H be the direct sum of countably infinite many copies of the cyclic group of order 2. Define the group $N := \bigoplus_{h\in H} H$. Of course, H acts on N by permuting indices in the direct sum, so we can form the semi-direct product $G := N \rtimes H$. The group G we refer to as the Leptin-Hulanicki group. The group G is defined in [**71**, Section 3], but Hulanicki attributes the example to Leptin, hence we term this group as the *Leptin-Hulanicki group.* This group is class 2 solvable and it is shown in [**71**] that the group G is not Hermitian.

3.8.5. Horocyclic groups of automorphism of trees and scale groups.

Let $d \in \mathbb{N}_{>2}$ and let $T_d = (VT_d, ET_d)$ denote the regular tree of degree d, that is, the infinite tree with the property that every vertex has degree d. We denote the boundary of T_d by ∂T_d. For $\gamma \in \partial T_d$, define two groups by $G_\gamma := \{x \in \mathrm{Aut}(T_d) : x(\gamma) = \gamma\}$ and $B_\gamma := \{x \in G_\gamma : \exists v \in VT_d, x(v) = v\}$.

Proposition 3.65. *The group B_γ is locally elliptic and $G_\gamma \cong B_\gamma \rtimes \mathbb{Z}$.*

Proof. This is a standard result. See, for example, [**51, 152**]. ☐

A closed vertex transitive subgroup $G \leqslant G_\gamma$ is called a ***scale group***. Scale groups are studied in [**6, 66, 152**] and it is shown they have important connections with the structure theory of totally disconnected locally compact groups. For example, every non-uniscalar totally disconnected locally compact group has a subquotient isomorphic to a scale group. Also, these groups are simultaneously analogues of parabolic subgroups and $ax+b$ groups in the theory of tdlc groups.

Every scale group is of the form $G = N \rtimes \mathbb{Z}$ with N locally elliptic, so understanding the harmonic analysis of locally elliptic groups is critical to understanding the harmonic analysis of scale groups.

The following is an open question in the harmonic analysis of tdlc groups.

Question 3.66. Let $G = N \rtimes \mathbb{Z}$ be a scale group. Under what assumptions is G Hermitian?

As far as the author is aware, it is even an open question as to whether $\mathbb{Q}_p \rtimes \mathbb{Z}$ is Hermitian (*c.f.* [**113**, Section 3.6]).

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CHAPTER 4

The type I dichotomy for two-step nilpotent groups

with Pierre-Emmanuel Caprace, arXiv:2509.26212.

Abstract

We address the type I dichotomy for two-step nilpotent locally compact groups. Invoking work of Baggett–Kleppner, we characterize the closed points of the unitary dual of such a group G purely in terms of the group structure. An algebraic criterion characterizing when G is a type I group is derived. We show that this criterion automatically holds if G is a central extension of vector groups over a non-discrete locally compact field k such that the commutator map is k-bilinear. As an application, we show that the unipotent radicals of minimal parabolics in simple algebraic groups of k-rank one are type I groups. We also discuss the type I dichotomy for p-torsion contraction groups, and exhibit, for each prime p, uncountably many pairwise non-isomorphic such groups that are not type I. This answers a recently posed question by the second author. Finally, we adapt a recent construction of Chirvasitu to obtain numerous examples of two-step nilpotent torsion locally compact groups that are not type I, but that embed as closed cocompact normal subgroups in two-step nilpotent groups that are type I.

4.1. Introduction

A topological group G is called **type** I if every unitary representation of G generates a type I von Neumann algebra [**42**, §5]. The classical theory of unitary representations of locally compact groups shows that there is a strong dichotomy between type I groups and those which are not type I, that we shall henceforth call the **type** I **dichotomy**: type I groups are precisely those groups all of whose unitary representations decompose *uniquely* into a direct integral of irreducible unitary representations [**42**, §8]. Kirillov suggested to use the terms **tame** vs. **wild** instead of type I vs. non-type I, see [**7**, §8.F.b, Comment (13)]. Also, a result of Glimm, referred to as *Glimm's theorem* (see [**52**, Theorem 7.6] or [**57**]), implies that a second countable locally compact group G is type I if and only if its unitary dual, $\widehat{G}$, endowed with Fell's topology, is a T_0 topological space. Another natural, and formally stronger condition, is that the unitary dual $\widehat{G}$ be a T_1 space. In that case, the group G is called **CCR** (see [**52**, §7.2]).

It is thus an important problem to determine which locally compact groups satisfy the type I property or the stronger CCR property. Classically, it was Lie groups and algebraic groups over local fields where this problem received the most attention, and for many of these groups, the solution to this problem is well understood [**76, 146, 147, 90, 8**]. One notable exception, pointed out by Bekka–Echterhoff [**8**, Remark 1(iv)], is the class of unipotent algebraic groups over local fields of positive characteristic: it is currently unknown whether all these groups are of type I.

In this paper, we address the type I dichotomy for **two-step nilpotent groups**, i.e. locally compact groups whose commutator subgroup is central (equivalently, they are nilpotent of nilpotency class at most 2). Classical results due to J. Dixmier [**40**] and A. Kirillov [**76**] ensure that every connected nilpotent Lie group is CCR, hence of type I. On the other hand, Thoma's theorem ensures that a discrete group is of type I if and only if it is virtually abelian (in particular every discrete group of type I is CCR), see [**141**] or [**7**, Theorem 7.D.1]. In particular, there exist two-step nilpotent groups that are not of type I, e.g. the Heisenberg group over $\mathbb{Z}$.

Since the type I and CCR properties are characterized by a separation condition on the unitary dual, it is a natural question to investigate the closed points in $\widehat{G}$. We denote by $[\pi]$ the class of an irreducible representation with respect to unitary equivalence.

THEOREM 4.1 (See Theorem 4.15). *Let G be a two-step nilpotent second countable locally compact group G with center Z, and π an irreducible unitary representation of G with central character χ (see* [**7**, Definition 1.A.12]*). Then the following assertions are equivalent:*

- *(i) $[\pi]$ is a closed point in the unitary dual $\widehat{G}$;*
- *(ii) The homomorphism $\omega_\chi\colon G/Z \to \widehat{G/Z}$, defined by setting $\omega_\chi(gZ)(hZ) = \chi([g,h])$, has a closed image.*

The proof relies on a reformulation of results due to Baggett–Kleppner [**5**] concerning projective representations of abelian groups. We also rely on the so-called Poguntke parametrization of the primitive dual of G, that will be recalled in Theorem 4.12 below. The following characterization of the type I property purely in terms of the algebraic/topological structure of G follows readily.

Corollary 4.2 (See Corollary 4.17)**.** *For a two-step nilpotent second countable locally compact group G with center Z, the following conditions are equivalent:*

- *(i) G is type* I*;*

(ii) G is CCR;
(iii) For each $\chi \in \widehat{Z}$, the homomorphism $\omega_\chi \colon G/Z \to \widehat{G/Z}$ has closed image.

The equivalence between (i) and (ii) follows directly from a general fact, due to Poguntke, that the primitive dual of a two-step nilpotent locally compact group is a T_1 space (see Corollary 4.13 below).

Several variants of the closedness condition in Theorem 4.1(ii) will be established for totally disconnected groups, see Proposition 4.19. This yields the following consequence for discrete countable groups that is of independent interest.

Corollary 4.3 (See Corollary 4.24)**.** *For a two-step nilpotent countable group G with center Z, the following conditions are equivalent:*

(i) G is type I*;*
(ii) G is virtually abelian;
(iii) For each character $\chi \in \widehat{Z}$, the quotient $G/\ker(\chi)$ is center-by-finite.

The equivalence between (i) and (ii) is valid for all discrete groups by virtue of Thoma's theorem mentioned above. The equivalence between (ii) and (iii) is a purely algebraic statement that will be useful in Section 4.3.

Corollary 4.2 allows us to recover, in a unified way, results on connected groups and on p-adic analytic groups, see Corollary 4.29. However, its main interest is that it also covers p-torsion groups, where p is an arbitrary prime. This will be concretized through the following statement.

THEOREM 4.4 (See Theorem 4.30). *Let k be a non-discrete locally compact field, and A, N be finite-dimensional k-vector spaces. Let G be a locally compact group that is a central extension of A by N. If the map $A \times A \to N$ induced by the commutator map on G is k-bilinear, then G is type* I.

It is important to keep in mind that the commutator map $A \times A \to N$ is automatically bi-additive. Theorem 4.4 allows us to establish, in a uniform fashion, that several natural classes of unipotent algebraic groups over local fields of arbitrary characteristic are type I.

Corollary 4.5. *Let k be a non-discrete locally compact field. The following locally compact two-step nilpotent groups are all type* I*:*

(1) The $2n+1$-dimensional Heisenberg group over k, for all $n \geqslant 1$.
(2) Given a two-step nilpotent Lie algebra $\mathfrak{g}$ over k with Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$, the unipotent group $E(\mathfrak{g})$ with underlying set $\mathfrak{g}$ and multiplication defined by $v.w = v + w + [v,w]_{\mathfrak{g}}$.

(3) The unipotent radical of a minimal k-parabolic subgroup in an absolutely simple algebraic k-group of k-rank one.

Remark 4.6.

(1) In view of Corollary 4.2, the item (1) in Corollary 4.5 recovers and strengthens [**27**, Theorem 5.15].
(2) The map $\mathfrak{g} \mapsto E(\mathfrak{g})$ in Corollary 4.5(2) is closely related to the so-called **Lazard correspondence**. The group $E(\mathfrak{g})$ is abelian if $\mathrm{char}(k) = 2$, but non-abelian if $\mathrm{char}(k) \neq 2$ and $\mathfrak{g}$ is non-commutative.
(3) The groups in Corollary 4.5(3) include non-abelian groups in all characteristics.

Remark 4.7. As mentioned above, it is currently unknown whether all unipotent algebraic groups over local fields of positive characteristic are type I, see [**8**, Remark 1(iv)]. An alternative approach to this problem is provided by the work of Echterhoff–Klüver [**46**], developing the Kirillov orbit method for rather general nilpotent groups, following pioneering work of R. Howe [**68**]. This approach has the advantage that it applies to nilpotent groups of higher nilpotency class c, and that it provides a description of the unitary dual for type I groups. However, there are several important technical restrictions. One of them is that the group should not contain p-torsion for any prime $p \leqslant c$; another one is a condition of *regularity* that is rather restrictive in positive characteristic[1].

The last part of this paper aims at providing additional explicit examples of two-step nilpotent groups illustrating the type I dichotomy. We focus on groups that are central extensions of the additive group $A = \mathbb{F}_p(\!(t)\!)$ by itself, where p is an arbitrary prime. We require that the commutation relations induce a map $A \times A \to A$ that sends pairs (t^a, t^b) to monomials (see Definition 4.32). We establish criteria for such groups to be type I (see Theorem 4.33) and non-type I (see Theorem 4.35). As an application, we obtain the following result concerning the family of two-step nilpotent groups $A \times_{\eta_s} A$ introduced by Glöckner–Willis [**59**, Section 8]. A group in that family is defined by a cocycle η_s determined by a sequence $s \colon \mathbb{N}_{>0} \to \{0,1\}$. The precise definition will be recalled in Section 4.3.3.

Theorem 4.8. *Let p be prime. Let $s \in \{0,1\}^{\mathbb{N}_{>0}}$ and $G = A \times_{\eta_s} A$ the group mentioned above.*

Then G is type I *as soon as the sequence s satisfies any of the following conditions for some integer $c \geqslant 0$.*

---

[1] In Example 8.2 of [**46**], it is claimed that if p is a prime and G is a nilpotent locally compact group of nilpotency class $c < p$ in which every element is contained in a pro-p subgroup, then the regularity condition automatically holds for G. That claim is inaccurate: indeed, in the additive group $\mathfrak{g} = \mathbb{F}_p(\!(t)\!)$, the sum of two closed subgroups can fail to be closed (e.g. the subgroups $\mathfrak{a} = \mathbb{F}_p[t^{-1}]$ and $\mathfrak{b} = \mathbb{F}_p[t^{-1} + t]$ are both discrete, and their sum is dense).

(1) $s(z) = 1$ *for all* z *with* $z > c$.

(2) $s(z) = 0$ *for all even* z, *and* $s(z) = 1$ *for all odd* z *with* $z > c$.

(3) There is an integer $d > 0$ *such that* $s(z) = 0$ *for all* $z \notin d\mathbb{N}$, *and* $s(z) = 1$ *for all* $z \in d\mathbb{N}$ *with* $z > c$.

On the other hand, G *is not type* I *if* s *satisfies the following.*

(4) For some integer $d > 0$, *we have* $s(d) = 1$ *and* $s(dn) = 0$ *for all* $n \geqslant 2$.

It is proved in [**27**, Theorem 5.9] that if s is non-zero and finitely supported, then G is not type I. Theorem 4.8(4) recovers and strengthens that fact.

The following consequence of Theorem 4.8 (see also Proposition 4.38) provides a positive solution to [**27**, Problem 5.13].

Corollary 4.9. *For a given prime* p, *the family of groups* $A \times_{\eta_s} A$ *contains infinitely many pairwise non-isomorphic type* I *groups, and uncountably many pairwise non-isomorphic non-type* I *groups.*

Remark 4.10. The sequences giving rise to type I groups in Theorem 4.8 are asymptotically periodic, but not asymptotically zero. However, the precise boundary between type I groups and non-type I groups remains unclear. Notice in particular that, according to Theorem 4.8, the group G is type I if $s = (1, 0, 1, 0, 1, 0, 1, \dots)$ but not type I if $s = (1, 1, 1, 0, 1, 0, 1, \dots)$.

The class of type I groups enjoys various stability properties with respect to basic algebraic constructions: e.g. passing to open subgroups, passing to quotient groups, forming finite direct products, see [**7**, Proposition 6.E.21]. It is moreover known that if G is a locally compact group admitting a closed normal subgroup N such that G/N is compact, then G is type I as soon as N is so. Recently, A. Chirvasitu [**33**, §2.2] has shown that the converse does not hold, even within the class of two-step nilpotent groups: in general the type I property need not be inherited by a closed cocompact normal subgroup. By revisiting Chirvasitu's construction in the light of the results of this paper, we establish the following. It applies notably to all the contraction groups appearing in Theorem 4.8, that are further discussed in Section 4.3.3.

THEOREM 4.11. *Let* G *be a two-step nilpotent second countable locally compact group with center* Z. *Let* $N \leqslant G$ *be a closed subgroup with* $[G, G] \leqslant N \leqslant Z$. *Let* p *be a prime, and suppose that* N *and* G/N *both have exponent* p.

Then for each character $\chi \in \widehat{N}$, *the quotient* $G/\ker(\chi)$ *continuously embeds as a closed cocompact normal subgroup in a two-step nilpotent locally compact group that is type* I.

For all non-type I groups arising in Theorem 4.8, there is a central character χ such that $G/\ker(\chi)$ is not type I (see Remark 4.40). Hence Theorem 4.11 yields a broad family of non-type I groups embedding as closed cocompact normal subgroups in two-step nilpotent locally compact groups.

While we have focused on second countable groups in this introduction, several results mentioned above are established below without any countability assumptions.

Layout of the article. In Section 4.2 we establish Theorem 4.1 and its consequences, and present various supplements for the class of totally disconnected groups. In Section 4.3, we focus on two-step nilpotent groups with monomial commutation relations, finishing with a proof of Theorem 4.8. Finally, the proof of Theorem 4.11 is given in Section 4.4.

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4.2. Unitary representation theory of two-step nilpotent groups

Throughout this paper, we define the commutator of two elements g, h of a group as $[g, h] = ghg^{-1}h^{-1}$. We assume that the reader has some familiarity with the theory of unitary representations of locally compact groups (see [**42**] and [**7**]) and with Pontryagin duality (see [**16**, Ch. II] and [**138**, Ch. F]).

4.2.1. Characterizing closed points in the unitary dual. Given an irreducible unitary representation π of a locally compact group G, we denote by $[\pi]$ its equivalence class under unitary equivalence. Thus $[\pi] \in \widehat{G}$. In the special case where π is one-dimensional, we identify $[\pi]$ with the character of π, which is a complex valued function on G. For each closed normal subgroup L of G containing the commutator subgroup $[G, G]$, we define $L^{\perp} \subseteq \widehat{G}$ as the closed subset consisting of those characters of G that are trivial on L. Note that $L^{\perp}$ is naturally isomorphic to $\widehat{G/L}$, and carries a canonical group structure defined by pointwise multiplication of characters. By Schur's lemma, the restriction of an irreducible representation π to the center Z of G defines an action of Z by scalar operators. Thus $\pi(z) = \chi_{\pi}(z)\mathrm{Id}$ for some character $\chi_{\pi} \in \widehat{Z}$, called the **central character** of π.

We start by recalling Poguntke's parametrization of the primitive ideal space $\mathrm{Prim}(G)$ of a two-step nilpotent group G, which appears as the first proposition in [**120**, Part I] and, according to Poguntke, relies on ideas due to R. Howe, see [**68**, Prop. 5] and Kaniuth [**73**, Lemma 2]. We follow the presentation from [**75**, §5.8].

So let G be a two-step nilpotent locally compact group with center Z. Every character $\chi \in \widehat{Z}$ defines a homomorphism

$$\omega_\chi \colon G \to Z^\perp : g \mapsto \big(xZ \mapsto \chi([g,x])\big). \tag{4.2.1}$$

Since ω_χ is trivial on Z and since $Z^\perp$ is canonically isomorphic to $\widehat{G/Z}$, we may as well view ω_χ as a homomorphism of abelian groups

$$\omega_\chi \colon G/Z \to \widehat{G/Z}.$$

Given $\chi \in \widehat{Z}$, we set $L_\chi = \{g \in G \mid [g,G] \subseteq \ker(\chi)\}$. We also set

$$A_\chi = \{\alpha \in \widehat{L_\chi} \mid \alpha|_Z = \chi\}.$$

Since $L_\chi/\ker(\chi)$ is abelian, and since characters of closed subgroups of locally compact abelian groups can always be extended (see [**16**, Chap. II, §1, no. 7, th. 4]), we infer that A_χ is non-empty. Set

$$\mathcal{P} = \{(\chi,\alpha) \mid \chi \in \widehat{Z},\ \alpha \in A_\chi\}.$$

The first part of the following result is known as **Poguntke's parametrization** of the primitive ideal space $\mathrm{Prim}(G) := \mathrm{Prim}(C^*(G))$.

THEOREM 4.12. *Let G be a two-step nilpotent locally compact group.*

(i) The map

$$(\chi,\alpha) \mapsto C^*\text{-}\ker(\mathrm{ind}_{L_\chi}^G \alpha)$$

establishes a one-to-one correspondence from $\mathcal{P}$ to $\mathrm{Prim}(G)$.

(ii) For all $(\chi_1,\alpha_1),(\chi_2,\alpha_2) \in \mathcal{P}$, the weak containment $\mathrm{ind}_{L_{\chi_1}}^G \alpha_1 \preceq \mathrm{ind}_{L_{\chi_2}}^G \alpha_2$ is equivalent to the equality $(\chi_1,\alpha_1) = (\chi_2,\alpha_2)$.

(iii) Let π be an irreducible unitary representation of G with central character χ. Then π is weakly equivalent to $\pi \otimes \psi$ for all $\psi \in L_\chi^\perp$. Moreover, the group $L_\chi^\perp$ coincides with the closure of the image of ω_χ.

PROOF. For (i), see [**75**, Theorem 5.66]. The item (ii) follows from the arguments in the last part of the proof of [**75**, Theorem 5.66]. The item (iii) is explicitly established in the same proof (see the last few lines on page 260 and the first few lines on page 261 in [**75**]). □

Corollary 4.13. *Let G be a two-step nilpotent locally compact group. Given irreducible unitary representations π_1, π_2 of G, we have $\pi_1 \preceq \pi_2$ if and only if $\pi_1 \sim \pi_2$. Equivalently, the space* $\mathrm{Prim}(G)$ *is* T_1.

In particular, a two-step nilpotent locally compact group is type I *if and only if it is CCR.*

PROOF. This follows directly from Theorem 4.12(i) and (ii). □

Remark 4.14. The fact that $\mathrm{Prim}(G)$ is a T_1-space is pointed out by D. Poguntke in the first paragraph of Part I in [**120**]. This property is also known for discrete nilpotent groups (see [**68**] for finitely generated groups and [**119**] for the general case) and for nilpotent locally compact groups containing an open normal subgroup that is compactly generated (see [**?**]). It is an intriguing open problem to determine whether it holds for an arbitrary second countable nilpotent locally compact group.

Recall that a unitary representation ρ of G is called a **factor representation** if the von Neumann algebra generated by $\rho(G)$ is a factor. In that case, the operator $\rho(z)$ is scalar for any element z of the center Z of G. Thus $\rho(z) = \chi_\rho(z)\mathrm{Id}$ for some character $\chi_\rho \in \widehat{Z}$, called the **central character** of ρ. This definition is consistent with the one given above: indeed every irreducible representation is a factor representation. Combining Theorem 4.12 with the work by Baggett–Kleppner [**5**] (valid without any second countability assumption), we establish the following characterization of the closed points in $\widehat{G}$.

Theorem 4.15. *Let G be a locally compact group that is two-step nilpotent, and π an irreducible unitary representation of G with central character χ. Then the following assertions satisfy the implications (i) $\Leftarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). If in addition G is second countable, then they are all equivalent, and the second sentence of (ii) can be discarded.*

- *(i)* *$[\pi]$ is a closed point in the unitary dual $\widehat{G}$.*
- *(ii)* *The homomorphism $\omega_\chi \colon G/Z \to \widehat{G/Z}$, defined as in Equation 4.2.1, has a closed image. Moreover, the image of every open subset is relatively open.*
- *(iii)* *Every factor representation of G with central character χ is type* I*.*
- *(iv)* *Every factor representation of G weakly equivalent to π is type* I*.*

Proof. We may view G as a central extension of the abelian group G/Z associated with a Z-valued 2-cocycle $\kappa \colon G/Z \times G/Z \to Z$. The composite map $\omega = \chi \circ \kappa$ defines a $\mathbf{C}$-valued cocycle, and there is a canonical one-to-one correspondence between the unitary representations of G with central character χ, and the so-called ω-representations of G/Z considered by Baggett–Kleppner [**5**] (see the first few lines on p. 302 in [**5**]).

The image of ω_χ is contained in $L_\chi^\perp$, which is a closed subset of $Z^\perp$. By the Corollary to Theorem 3.2 in [**5**], we infer that (ii) and (iii) are equivalent (see also the discussion preceding Theorem 3.3 in [**5**], showing that the second sentence of (ii) can be discarded if G is second countable).

Since restriction of representations preserves weak containment, every factor representation weakly contained in π must have the same central character as π. Thus it is clear that (iii) implies (iv).

Let us now show that (iii) implies (i). Let π be an irreducible unitary representation of G. Since $\mathrm{Prim}(G)$ is T_1 by Theorem 4.12, every primitive ideal in the C^*-algebra of G is maximal, hence the C^*-algebra generated by π, denoted by $C^*(\pi)$, is simple. Every factor representation of $C^*(\pi)$ naturally defines a factor representation of G that is weakly contained in π. Considering the restriction of such a factor representation to Z, we infer that its central character is χ since the operation of restriction preserves weak containment. If (iii) holds this factor representation is type I. From Glimm's theorem and its extension to the non-separable case by Sakai (see [**42**, Theorem 9.1] and the discussion in [**11**, IV.1.5.8]), we know that a C^*-algebra is type I if and only if each of its factor representations is type I. We deduce that the C^*-algebra $C^*(\pi)$ is type I. Since $C^*(\pi)$ is also simple, it must be *elementary*, i.e. it is isomorphic to the C^*-algebra of compact operators on a Hilbert space. It follows that $C^*(\pi)$ has a unique equivalence class of non-zero irreducible representations (see [**42**, Corollary 4.1.5]). In other words, every irreducible unitary representation of G weakly contained in π is equivalent to π. This proves that (iii) implies (i).

Let us now show that (iv) implies (ii). Indeed, assume that (ii) fails. By Theorem 4.12(i), there exists $\alpha \in A_\chi$ such that $\pi \sim \mathrm{ind}_{L_\chi}^{G}\alpha$ (since $\pi(\ell)$ is a scalar operator for each $\ell \in L_\chi$, this just means that $\alpha \in \widehat{L_\chi}$ is the associated character). We have $\ker(\pi) = \ker(\alpha) \leqslant L_\chi$, hence $\ker(\alpha)$ is normal in G and the representation π factors through $G/\ker(\alpha)$. The center of $G/\ker(\alpha)$ is $L/\ker(\alpha)$. Recall that ω_χ may be viewed as a homomorphism of G/L_χ to its dual. Using that (ii) is equivalent to (iii) for the quotient group $G/\ker(\alpha)$, we deduce that some factor representation ρ of $G/\ker(\alpha)$ with central character α is not type I. By precomposing ρ with the canonical projection $G \to G/\ker(\alpha)$, we may view ρ as a factor representation of G. By definition, for each $\ell \in L_\chi$ we have $\rho(\ell) = \alpha(\ell)\mathrm{Id}$. This implies that each irreducible representation $\sigma \preceq \rho$ has the same Ponguntke parameters as π. Therefore, we have $\sigma \sim \pi$ by Theorem 4.12. It follows that the factor representation ρ is weakly equivalent to π. Since ρ is not type I, we deduce that (iv) indeed fails.

It remains to show that (i) implies (ii) under the additional hypothesis that G is second countable. If (iv) fails, then G has a factor representation ρ weakly equivalent to π, that is not type I. Invoking [**7**, Corollary 7.F.4], it follows that G has uncountably many inequivalent irreducible representations weakly equivalent to ρ. Each of them is weakly equivalent to π, so that the closure $\overline{\{[\pi]\}}$ contains uncountably many points. Thus (i) fails. $\square$

Remark 4.16. In order to prove the implication (i) $\Rightarrow$ (ii) in Theorem 4.15 without the second countability assumption, we would need to know that if the simple C^*-algebra $C^*(\pi)$ has a unique equivalence class of irreducible representation, then it is type I (hence elementary). This is not true for non-separable simple C^*-algebras in general (see the counterexample to Naimark's problem in [**1**]). We do not know whether this property holds for the family of simple C^*-algebras $C^*(\pi)$ arising in Theorem 4.15.

The following characterization of two-step nilpotent groups of type I follows easily.

Corollary 4.17. *For a two-step nilpotent locally compact group G with center Z, the following conditions are equivalent.*

- *(i) G is type* I.
- *(ii) G is CCR.*
- *(iii) For each $\chi \in \widehat{Z}$, the homomorphism $\omega_\chi \colon G/Z \to \widehat{G/Z}$ has closed image. Moreover, the image of every open subset is relatively open.*

If in addition G is second countable, then this is also equivalent to:

- *(iv) For each $\chi \in \widehat{Z}$, the homomorphism $\omega_\chi \colon G/Z \to \widehat{G/Z}$ has closed image.*

Proof. The equivalence between (i) and (ii) was recorded in Corollary 4.13. The equivalence between (i), (iii) and (iv) follows from the equivalence between (ii) and (iii) in Theorem 4.15 (since a group is type I if and only if each of its factor representations is type I, see [**42**, Theorem 9.1] and the discussion in [**11**, IV.1.5.8]). □

Corollary 4.17 provides a characterization of the type I condition in terms of the topological/algebraic structure of G. Various applications will be presented in Section 4.2.3 below. Let us already record the following.

Corollary 4.18. *Let G be a two-step nilpotent locally compact group, and N be a closed subgroup contained in the center of G. Assume that N splits as the direct product of two closed subgroups N_1 and N_2 such that $[G, G] \subseteq N_2$. Then G is type* I *if and only if G/N_1 is type* I.

Proof. By definition, the type I property passes to quotient groups, so G/N_1 is type I as soon as G is.

Let Z be the center of G. Since $[G, G] \leqslant N \leqslant Z$ by hypothesis, we have $Z^\perp \subseteq N^\perp$. Moreover for each $\chi \in \widehat{Z}$, the homomorphism ω_χ is trivial on Z-cosets, hence also on N-cosets. Thus we may naturally view ω_χ as a map from G/N to its dual. By hypothesis we have $N \cong N_1 \times N_2$. Thus $\widehat{N} \cong \widehat{N_1} \times \widehat{N_2}$. Hence there exists $\chi' \in N_1^\perp$, such that χ and χ' have the same restriction on N_2. It follows that ω_χ and $\omega_{\chi'}$ have the same image in $\widehat{G/N}$. Assuming that G/N_1 is type I, it follows from Corollary 4.17 that the image of $\omega_{\chi'}$ in $\widehat{(G/N_1)/(N/N_1)} \cong \widehat{G/N}$ is closed. The result follows. □

4.2.2. Totally disconnected groups.

We shall now focus on totally disconnected locally compact groups (whose name will henceforth be abbreviated by **tdlc groups**). In that case, additional algebraic characterizations may be added to Theorem 4.15 and Corollary 4.17.

Recall that every continuous homomorphism of a tdlc group to a Lie group has an open kernel, since Lie groups have no small subgroups. In particular, every character of a tdlc group has an open kernel. It follows that if G is a two-step nilpotent tdlc group with center Z and $\chi \in \widehat{Z}$, then there exists a compact open subgroup $U \leqslant G$ with $U \cap Z \leqslant \ker(\chi)$. We infer that the derived group $[U, U]$ is contained in $\ker(\chi)$.

Proposition 4.19. *Let G be a second countable two-step nilpotent tdlc group with center Z, and let $\chi \in \widehat{Z}$. Let $\omega_\chi \colon G/Z \to \widehat{G/Z}$ be the homomorphism defined in Equation 4.2.1. Let also $U \leqslant G$ be a compact open subgroup such that $[U, U] \leqslant \ker(\chi)$. Define*

$$O_{\chi,U} = \{g \in G \mid [g, U] \subseteq \ker(\chi)\}$$

and

$$L_{\chi,U} = \{g \in O_{\chi,U} \mid [g, O_{\chi,U}] \subseteq \ker(\chi)\}.$$

Let $\tilde{\chi}$ be a character of $L_{\chi,U}$ that extends χ (the quotient $L_{\chi,U}/\ker(\chi)$ is an abelian group that contains $Z/\ker(\chi)$ as a closed subgroup, hence such a character $\tilde{\chi}$ always exists by [**16**, Chap. II, §1, no. 7, th. 4]*).*

The following assertions are equivalent.

- *(i) The image of $\omega_\chi \colon G/Z \to \widehat{G/Z}$ is closed.*
- *(ii) The quotient $O_{\chi,U}/L_{\chi,U}$ is finite.*
- *(iii) The quotient $O_{\chi,U}/\ker(\chi)$ is **center-by-finite**, i.e. its center is of finite index.*
- *(iv) The quotient $O_{\chi,U}/\ker(\chi)$ is a type* I *group.*
- *(v) The quotient $O_{\chi,U}/\ker(\tilde{\chi})$ is a type* I *group.*
- *(vi) The quotient $O_{\chi,U}/\ker(\tilde{\chi})$ is center-by-finite.*

PROOF. By definition, the quotient group $L_{\chi,U}/\ker(\chi)$ is the center of $O_{\chi,U}/\ker(\chi)$. The equivalence between (ii) and (iii) readily follows.

Recall that a subgroup of a locally compact group is closed if and only if it is locally closed (see [**17**, Ch. III, § 2., Prop. 4]). In the dual $\widehat{G/Z}$, an identity neighbourhood is provided by all the characters that vanish on UZ/Z. Given $g \in G$, the character $\omega_\chi(gZ)$ vanishes on UZ/Z if and only if $[g, U] \leqslant \ker(\chi)$, which means that $g \in O_{\chi,U}$. This proves that ω_χ has a closed image if and only if the restriction of ω_χ to $O_{\chi,U}$ has a closed image.

Now we invoke Theorem 4.12(iii) that we apply to the group $O_{\chi,U}/\ker(\chi)$ and the character $\tilde{\chi}$ defined on its center. By definition, we have

$$L_{\tilde{\chi}} = \{g \in O_{\chi,U}/\ker(\chi) \mid [g, O_{\chi,U}] \subseteq \ker(\tilde{\chi})\}.$$

Since $[G,G] \leqslant Z$, we infer that $L_{\tilde{\chi}} = L_{\chi,U}/\ker(\chi)$ and that $\omega_{\tilde{\chi}}$ is nothing but the restriction of ω_χ to $O_{\chi,U}$ (which is indeed trivial on $\ker(\chi)$). The map $\omega_{\tilde{\chi}}$ may be viewed as a map

$$\omega_{\tilde{\chi}} \colon O_{\chi,U}/L_{\chi,U} \to \widehat{O_{\chi,U}/L_{\chi,U}},$$

whose image is dense by Theorem 4.12(iii).

Since $L_{\chi,U}$ contains U, it is an open subgroup of $O_{\chi,U}$. The image of ω_χ is thus a countable dense subgroup of the compact group $\widehat{O_{\chi,U}/L_{\chi,U}}$. We infer that $\omega_{\tilde{\chi}}$ has a closed image if and only if the countable discrete group $O_{\chi,U}/L_{\chi,U}$ is compact. The equivalence between (i) and (ii) follows.

It is clear that (iii) implies (iv). Since the type I conditions passes to quotient groups, and since $\ker(\tilde{\chi})$ contains $\ker(\chi)$, the assertion (iv) implies (v).

Finally, we observe that $L_{\chi,U}/\ker(\tilde{\chi})$ is the center of $O_{\chi,U}/\ker(\tilde{\chi})$. Hence (iii) and (vi) are equivalent. If (v) holds, then we invoke Corollary 4.17 for the group $O_{\chi,U}/\ker(\tilde{\chi})$ and the central character $\tilde{\chi}$ (viewed as a character defined on the quotient $L_{\chi,U}/\ker(\tilde{\chi})$). We infer that the image of the homomorphism $\omega_{\tilde{\chi}}$ is closed. As seen above, this implies that $O_{\chi,U}/L_{\chi,U}$ is finite. Thus (v) implies (iii). ☐

Remark 4.20. The subgroup $O_{\chi,U}$ is open in G since it contains U, and $O_{\chi,U}/\ker(\chi)$ has an open center, namely $L_{\chi,U}/\ker(\chi)$. Hence the kernel $\ker(\tilde{\chi})$ is also open and so the quotient $O_{\chi,U}/\ker(\tilde{\chi})$ is always discrete.

Remark 4.21. Proposition 4.19 may be viewed as a manifestation of Mackey's little group method: the unitary representation theory of the larger group G is controlled by that of the subgroup $O_{\chi,U}$, which plays the role of the little group.

One may actually describe much more precisely how Mackey's little group method applies to the situation treated by Proposition 4.19. Indeed, the group $UZ/\ker(\chi)$ is an abelian closed normal subgroup of $G/\ker(\chi)$, and one may verify that it is regularly embedded. Moreover given a character ψ of $UZ/\ker(\chi)$, it is easy to see that its stabilizer G_ψ (for the natural G-action on the dual of $UZ/\ker(\chi)$) coincides with $O_{\chi,U}$. In view of Theorem 4.15, one can establish Proposition 4.19 by invoking Mackey's theorem [**100**, Theorem 3.11].

Remark 4.22. The items (ii) and (iii) in Proposition 4.19 depend on the choice of a compact open subgroup U with $[U,U] \leqslant \ker(\chi)$, while the item (i) does not. Hence the validity (ii) and (iii) for one specific subgroup U implies it for all.

Fix a descending chain $U = U_0 \geqslant U_1 \geqslant \dots$ of compact open subgroup that form a basis of identity neighbourhoods in G. For each $n \geqslant 0$, set $O_n = O_{\chi,U_n}$. The groups

(O_n) form an ascending chain of open subgroups. Given $g \in G$, the character $\omega_\chi(g)$ is continuous, hence it has an open kernel. It follows that $\ker(\omega_\chi(g))$ contains U_n for some n. Hence $g \in O_n$. Therefore we have $G = \bigcup_n O_n$. We have just seen that $O_0/\ker(\chi)$ is center-by-finite if and only if $O_n/\ker(\chi)$ is center-by-finite for all n. Thus (iii) implies that $G/\ker(\chi)$ is a countable ascending union of open subgroups that are center-by-finite.

Remark 4.23. In the case where G is discrete, one may take $U = \{e\}$. In that case $O_{\chi,U} = G$ and $L_{\chi,U} = L_\chi$. Thus, if G is discrete and countable, then the assertions of Theorem 4.15 hold if and only if G/L_χ is finite, if and only if $G/\ker(\chi)$ is center-by-finite.

Combining Proposition 4.19 with the results of the previous section, we obtain the following.

Corollary 4.24. *Let G be a countable two-step nilpotent group with center Z. Then the following conditions are equivalent.*

(i) *G is type* I.
(ii) *G is virtually abelian.*
(iii) *For each character $\chi \in \widehat{Z}$, the quotient $G/\ker(\chi)$ is center-by-finite.*

Proof. The equivalence between (i) and (ii) is valid for all discrete groups, according to Thoma's theorem [**141**, Satz 4]. The equivalence with (iii) is a consequence of Corollary 4.17 and Proposition 4.19 (see Remark 4.23). ☐

Remark 4.25. A finitely generated nilpotent group is virtually abelian if and only if it is center-by-finite. This follows for example from Bass–Guivrac'h's theorem describing the asymptotic growth type of a finitely generated nilpotent group (see [**44**, Theorem 14.26]). Thus the equivalence between (ii) and (iii) is obvious for finitely generated groups. However, a virtually abelian two-step nilpotent group that is not finitely generated may fail to be center-by-finite. As an example, let k be a field of prime order p. Consider the action of the additive group k on the 2-dimensional vector space over k, denoted by V, where $x \in k$ acts as the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Let A be the countable abelian group defined as the countably infinite direct sum of copies of V, and let $G = A \rtimes k$ be the semi-direct product, where k acts on each copy of V through the action defined above. Then G is virtually abelian and two-step nilpotent, but the center of G is of infinite index.

We mention a purely algebraic consequence of Corollary 4.24 that we will need later. It can be compared to the fact that an infinite direct sum of non-abelian groups is never virtually abelian.

Corollary 4.26. *Let G be a group and for each integer $n \geqslant 0$, let G_n be a non-abelian two-step nilpotent subgroup of G. Suppose that $[G_m, G_n] = \{e\}$ for all $m \neq n$. Set $A_n = [G_n, G_n]$. If the subgroup generated by $\bigcup_n A_n$ is isomorphic to the direct sum $\bigoplus_n A_n$, then G is not virtually abelian.*

Proof. Let $g_n, h_n \in G_n$ be non-commuting elements. Let $G_n^1 = \langle g_n, h_n \rangle$, $A_n^1 = [G_n^1, G_n^1]$ and $G^1 = \langle \bigcup_n G_n^1 \rangle$. It suffices to show that G^1 is not virtually abelian. Since the G_n's pairwise commute, the multiplication map defines a surjective homomorphism $\bigoplus_n G_n^1 \to G^1$. Since G_n^1 is two-step nilpotent for all n, it follows that G^1 is a countable two-step nilpotent group.

Set $A^1 = \langle \bigcup_n A_n^1 \rangle$ and $a_n = [g_n, h_n]$. The hypotheses imply that A^1 is abelian and splits as the direct sum of the A_n^1's. Hence its dual is the direct product $\prod_n \widehat{A_n^1}$. We infer that there is a character χ of A^1 such that $\chi(a_n) \neq 1$ for all n. We extend χ to a character defined on the center of G^1, that we also denote by χ. We claim that $G^1/\ker(\chi)$ is not center-by-finite.

Let Z be the center of $G^1/\ker(\chi)$ and suppose for a contradiction that Z has finite index. Let $\pi \colon G^1 \to G^1/\ker(\chi)$ be the canonical projection. By the pigeonhole principle, there is a strictly increasing function $\psi \colon \mathbb{N} \to \mathbb{N}$ such that the map $n \mapsto \pi(g_{\psi(n)})Z$ is constant. Thus $\pi(g_{\psi(0)}^{-1} g_{\psi(n)}) \in Z$. In particular, for all $n > 0$, the commutator

$$[\pi(g_{\psi(0)}^{-1} g_{\psi(n)}), \pi(h_{\psi(n)})] = \pi([g_{\psi(n)}, h_{\psi(n)}])$$

is trivial. This means that $[g_{\psi(n)}, h_{\psi(n)}] = a_{\psi(n)} \in \ker(\chi)$ for all $n > 0$. This contradicts the definition of χ.

In view of Corollary 4.24, we deduce that G^1 is not virtually abelian. The conclusion follows. □

Corollary 4.27. *Let G be second countable two-step nilpotent tdlc group. Then G is type* I *if and only if every discrete quotient of every open subgroup of G is type* I.

Proof. The type I condition passes to quotient groups and to open subgroups (see [**7**, Proposition 6.E.21(1)]). Thus the 'only if' part of the corollary holds.

The converse follows by combining Corollary 4.17, Proposition 4.19 and Remark 4.20. □

Corollary 4.27 may be combined with Corollary 4.24. We infer that for a second countable two-step nilpotent tdlc group, the type I condition is characterized by a purely algebraic conditions to be satisfied by discrete quotients of open subgroups.

4.2.3. Groups with a bilinear commutator map. Our next goal is to apply the previous results in order to show that several naturally occurring two-step nilpotent tdlc groups are type I.

We start with a general discussion. Let G be a group with center Z. Then G is two-step nilpotent if and only if the commutator group $[G, G]$ is contained in Z. Suppose that this the case. Given any subgroup N of G with $[G, G] \leqslant N \leqslant Z$, we may view G as a central extension of the abelian quotient $A = G/N$ by N. As such, it is described by the cohomology class of a 2-cocycle ω, which is defined as a map $\omega\colon A \times A \to N$ satisfying the following cocycle identity for all $a, b, c \in A$:

$$\omega(a + b, c) + \omega(a, b) = \omega(a, b + c) + \omega(b, c),$$

where we have used an additive notation for the group laws of abelian groups A and N. We shall focus on the case where G is locally compact and N is closed; the cocycle ω is then continuous.

Keeping in mind Theorem 4.15 and its consequences established above, we see that the type I property of G does not formally depend on the isomorphism type of G, or of the cohomology class of ω, but rather on the commutator map

$$G \times G \to [G, G] : (g, h) \mapsto ghg^{-1}h^{-1}.$$

This map is constant on cosets of Z, hence on cosets of N. Setting $A = G/N$ as above, we see that it induces canonically a map

$$\gamma\colon A \times A \to N$$

that we also call the **commutator map**.

Remark 4.28. It is important to observe that any biadditive map $A \times A \to N$ satisfies the cocycle identity. Since the commutator map on a two-step nilpotent group is biadditive, it follows that it can be viewed as a 2-cocycle. Thus every two-step nilpotent group G defines another two-step nilpotent group $\widetilde{G}$, with the same underlying set as G, and whose defining 2-cocycle is the commutator map of G. By considering the dihedral group of order 8 and the quaternion group of order 8, it is easy to see that two non-isomorphic groups may share the same commutator map, and that G need not be isomorphic to $\widetilde{G}$. If the map $a \mapsto 2a$ defines an automorphism of N, then $\widetilde{G}$ is isomorphic to $\widetilde{(\widetilde{G})}$, see [**36**, Proposition 2.8]. We do not need those facts, but we find it relevant to record that the commutator map captures enough information to determine whether G is type I, but not enough information a priori to determine the isomorphism type of G.

Let us start by observing that Corollary 4.17 recovers a couple of well-known cases of type I nilpotent groups. The fact that connected nilpotent locally compact groups are type I is due to Dixmier [**40**]. For p-adic groups, the corresponding result follows from [**48**] (see [**105**, §4] for further details).

Corollary 4.29. *Let G be two-step nilpotent locally compact group with center Z, and N be a closed subgroup with $[G, G] \leqslant N \leqslant Z$. Then G is type* I *in each of the following cases.*

(1) G/N is almost connected (i.e. the group of components of G/N is compact).
(2) G/N has a finite index open subgroup isomorphic to $\mathbb{Q}_p^d \times \mathbb{Z}_p^e$ for some prime p and some integers $d, e \geqslant 0$.

Proof. Set $A = G/N$. Let $\chi \in \widehat{Z}$ be a character. In view of Corollary 4.17, we must show that the image of the map $\omega_\chi \colon G \to Z^\perp$ from Equation 4.2.1 is closed. Since $[G, G] \leqslant N \leqslant Z$ by hypothesis, we have $Z^\perp \subseteq N^\perp$. Moreover ω_χ is trivial on Z-cosets, hence also on N-cosets. Thus we may naturally view ω_χ as a map from $A = G/N$ to its dual $\widehat{A}$.

Let A° be the identity component of A. By [**138**, Theorem 23.11], the group A° splits as the direct product $M \times V$, where M is the largest compact subgroup of A° and $V \cong \mathbb{R}^d$ is a vector group for some $d \geqslant 0$. Thus V is a subgroup of A and the quotient A/V is compact. It follows from Pontryagin duality that the dual $\widehat{A}$ splits as the direct product $\widehat{M} \times V^\perp$, see [**138**, Theorem 24.10]. Since $\mathbb{R}$ is self-dual, so is V, hence $\widehat{V} \cong V$. Moreover $V^\perp$ is isomorphic to $\widehat{A/V}$, which is discrete since A/V is compact.

In order to show that the continuous homomorphism $\omega_\chi \colon A \to \widehat{A}$ has a closed image, it suffices to show that its restriction to V has a closed image. Since V is connected, the restriction of ω_χ to V yields a continuous homomorphism $V \to \widehat{V} \cong V$. Since those groups are uniquely divisible, any group homomorphism is a morphism of $\mathbb{Q}$-modules, hence an $\mathbb{R}$-linear map by continuity (see [**138**, Theorem 24.6]). In particular its image is a vector subspace, hence it is closed.

In case $A = G/N$ is virtually a p-adic group of the form $A_0 = \mathbb{Q}_p^d \times \mathbb{Z}_p^e$, the proof is similar. Indeed it suffices to show that the restriction of ω_χ to the vector subgroup $\mathbb{Q}_p^d$ has a closed image. Since $\mathbb{Q}_p$ is self-dual, the dual of A_0 is isomorphic to $\mathbb{Q}_p^d \times (C_p^\infty)^e$, where C_p^∞ is the dual of $\mathbb{Z}_p$, which is isomorphic to group of p-power roots of unity. Since $\mathbb{Q}_p^d$ is divisible while C_p^∞ has no non-trivial divisible element, it follows that the restriction of ω_χ yields a continuous homomorphism of $\mathbb{Q}_p^d$ to itself. As in the case of $\mathbb{R}$, any such homomorphism is a morphism of $\mathbb{Q}$-modules by divisibility, hence an $\mathbb{Q}_p$-linear map by continuity. Thus the restriction of ω_χ to the vector subgroup $\mathbb{Q}_p^d$ has a closed image, and the result follows since $\mathbb{Q}_p^d$ is cocompact in A. $\square$

The following result will allow us to cover broader families of examples including algebraic groups over fields of positive characteristic.

THEOREM 4.30. *Let k be a non-discrete locally compact field. Let G be two-step nilpotent locally compact group with center Z, and N be a closed subgroup with $[G, G] \leqslant N \leqslant Z$. Suppose that the abelian groups $A = G/N$ and N are the additive groups of vector spaces over k.*

If the commutator map $\gamma\colon A \times A \to N$ is k-bilinear, then G is type I.

PROOF. Since A and N are locally compact, they are finite-dimensional over k. In particular G is second countable.

Let $\chi \in \widehat{Z}$ be a character. In view of Corollary 4.17, we must show that the image of the map $\omega_\chi\colon G \to Z^\perp$ from Equation 4.2.1 is closed. Since $[G, G] \leqslant N \leqslant Z$ by hypothesis, we have $Z^\perp \subseteq N^\perp$. Moreover ω_χ is trivial on Z-cosets, hence also on N-cosets. Thus we may naturally view ω_χ as a map from $A = G/N$ to its dual $\widehat{A}$.

The dual $\widehat{A}$ is naturally a k-vector space, with scalar multiplication defined by $\lambda\psi(a) = \psi(\lambda a)$ for any $\psi \in \widehat{A}$, $\lambda \in k$ and $a \in A$. The locally compact field k, hence also the group A, are isomorphic to their dual (see Theorem 3 on p. 40 in [**148**]). Now, using the k-bilinearity of γ, we observe that

$$\begin{aligned} \lambda\omega_\chi(g)(a) &= \chi \circ \gamma(g, \lambda a) \\ &= \chi \circ \gamma(\lambda g, a) \\ &= \omega_\chi(\lambda g)(a) \end{aligned}$$

for all $\lambda \in k$ and $g, a \in N$. Thus the image of ω_χ is invariant under scalar multiplication. Since ω_χ is a group homomorphism, its image is also stable under addition. Therefore, the image of ω_χ is a vector subspace, hence the zero set of a family of k-linear forms. Therefore, it is closed. □

We shall now established Corollary 4.5 stated in the introduction. In the case (3), the proof uses the notion of a **Moufang set**. We refer to [**39**] for an introduction to this topic. We do not repeat the detailed definitions, some of which are rather technical, but we only describe the key points from which the desired conclusion follows. Let us moreover note that special cases of Moufang sets of skew-hermitian type appearing below are those associated with a separable quadratic extension of k, defined in [**39**, Def. 5.4.1)]. The latter suffice to treat the unipotent radical of minimal parabolics in the unitary groups SU_3.

PROOF OF COROLLARY 4.5. We must show that the hypotheses of Theorem 4.30 are satsified.

For the groups satisfying (1) or (2), this follows from the definitions.

For the groups satisfying (3), the verification is more involved. Clearly we may assume that the group is non-abelian. Moreover, the case where $k = \mathbb{R}$ or $\mathbb{C}$ being covered by Corollary 4.29, we may further assume that k is a non-Archimedean local field (see [**148**, Chapter 1]).

Let us first observe that such a group U can indeed be viewed as a central extension of vector groups over k: indeed, this follows from [**15**, §3.17]. To check the bilinearity of the commutator map, we invoke the classification of the simple algebraic k-groups of k-rank one. Since k is a non-Archimedean local field, all such groups are of classical type (see the tables in [**142**, §4]). Moreover, by [**38**, Remark 4.5] (see also [**12**, Corollary 1.4.5]), the classification implies that U may be viewed as the root group of a Moufang set of skew-hermitian type, as defined in [**38**, §4.1]. Such a Moufang set is determined by a pseudo-quadratic space (D, D_0, σ, V, p) as defined in [**143**, (11.17)]. By definition, this means that D is a skew-field with center K which, in our case, is finite-dimensional over K, σ is an involution of D, D_0 is a K-subspace of D containing 1, such that $a^\sigma D_0 a \subseteq D_0$ for all $a \in D_0$, V is a right D-module, and p is a pseudo-quadratic form with corresponding skew-hermitian form h. Moreover, the defining ground field k coincides with the fixed field $k = \mathrm{Fix}_K(\sigma)$ of σ, which is of codimension 1 or 2 in K.

The group U is defined as in [**143**, (11.24)]. It is the set

$$U = \{(v, a) \in V \times D \mid p(v) - a \in D_0\}$$

with group multiplication

$$(v, a).(w, b) = (v + w, a + b + h(w, v)).$$

We define the closed normal subgroup N as $\{(0, a) \in U \mid a \in D_0\}$. Observe that $N \cong D_0$ is a K-vector group, hence a k-vector group. One checks that U/N is also a k-vector group with scalar multiplication induced by $\lambda.(v, a) = (v\lambda, a\lambda^2)$. Moreover, one computes that the commutator map is

$$\begin{aligned} \gamma\colon \qquad & U/N \times U/N \to D_0 \\ & \big((v, a)N, (w, b)N\big) \mapsto h(w, v) - h(v, w) = h(w, v) + h(w, v)^\sigma. \end{aligned}$$

This need not be bilinear over K, but it is bilinear over k. □

4.3. Contraction groups

4.3.1. Definition. In this section, we address the type I dichotomy for some tdlc contraction groups.

Let us start by recalling the definition. It is assumed throughout that automorphisms of topological groups are bicontinuous.

Definition 4.31. Let G be a topological group and $\alpha \in \mathrm{Aut}(G)$. The automorphism α is *contractive* if for all $g \in G$, $\alpha^n(g) \to \mathrm{id}_G$ as $n \to \infty$. A *contraction group* is a pair (G, α), where G is a topological group, and $\alpha \in \mathrm{Aut}(G)$ is contractive. If G satisfies a property P (e.g. locally compact, totally disconnected, torsion...) then we call (G, α) a P contraction group.

The structure theory of locally compact contraction groups is studied thoroughly in the articles [**136, 58, 59, 60**]. In view of the important role played by contraction groups within the structure theory of general tdlc groups, studying their representation theory is rather natural (see [**27**] for more information). Contraction groups also appear naturally in algebraic groups over local fields; indeed, unipotent radicals of parabolic subgroups are all contraction groups (see [**122**, Lemma 2.4]). In particular, the examples treated by Corollary 4.5(3) are all contraction groups.

4.3.2. Two-step nilpotent groups with monomial commutation relations.

We shall consider the family of two-step nilpotent groups defined as follows.

Definition 4.32. Let p be a prime, $\mathbb{F}_p$ be the finite field of order p, and$A = \mathbb{F}_p(\!(t)\!)$ the local field of formal Laurent series with coefficients in $\mathbb{F}_p$. Let G be a two-step nilpotent with center Z and N be a closed subgroup with $[G,G] \leqslant N \leqslant Z$, such that G/N and N are both isomorphic to the additive group A. Denote by $\gamma\colon A \times A \to A$ be the commutator map. We say that G has **monomial commutation relations** if there exists a map $u\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{F}_p$ such that

$$\gamma(t^m, t^n) = u(m,n)t^{m+n}$$

for all $m, n \in \mathbb{Z}$. Notice that such a group G is second countable, since A is so.

Since the commutator map is continuous and bi-additive, the identity $\gamma(t^m, t^n) = u(m,n)t^{m+n}$ completely determines the map γ.

In order for a group G as in Definition 4.32 to admit a contractive automorphism that preserves N, it is necessary that A has two contractive automorphisms, say α, β, such that $\gamma(\alpha(g), \alpha(h)) = \beta(\gamma(g,h))$ for all $g, h \in G$. The most natural contractive automorphisms are given by the multiplication by positive powers of t. If $\alpha\colon a \mapsto t^d a$ and $\beta\colon a \mapsto t^{2d}a$, we see that the map u from Definition 4.32 must satisfy

$$u(m+d, n+d) = u(m,n)$$

for all $m, n \in \mathbb{Z}$. This is clearly the case if $u(m,n) = \sigma_{m-n}$, where $\sigma\colon \mathbb{Z} \to \mathbb{F}_p : z \mapsto \sigma_z$ is a bi-infinite sequence of elements of $\mathbb{F}_p$. Since the map u satisfies $u(n,n) = 0$ and $u(m,n) = -u(n,m)$ for all m, n, we see that $\sigma_0 = 0$ and $\sigma_{-z} = -\sigma_z$ for all z. Thus σ is uniquely determined by its restriction to the positive integers. If G is as in Definition 4.32 and if moreover $u(m,n) = \sigma_{m-n}$, we say that G has **monomial commutation relations of type** σ.

Theorem 4.33. *Let G be a two-step nilpotent tdlc group with monomial commutation relations of type σ. Then G is type* I *if any of the following conditions are satisfied for some integer $c > 0$.*

(1) $\sigma_z \neq 0$ for all $z \in \mathbb{Z}$ with $|z| > c$.
(2) $\sigma_z = 0$ for all $z \in 2\mathbb{Z}$, and $\sigma_z \neq 0$ for all $z \in 2\mathbb{Z}+1$ with $|z| > c$.
(3) There is an integer $d > 0$ such that $\sigma_z = 0$ for all $z \notin d\mathbb{Z}$, and $\sigma_z \neq 0$ for all $z \in d\mathbb{Z}$ with $|z| > c$.

Proof. Since A is a group of exponent p, the image of every non-trivial character is a cyclic group of order p. We may thus view the Pontryagin dual $\widehat{A}$ of A as the group of continuous homomorphisms $\chi \colon A \to \mathbb{F}_p$.

In order to show that G is type I, we shall apply Proposition 4.19 for an arbitrary character $\chi \in \widehat{A}$. We may assume without loss of generality that χ is non-trivial. Set $a_m = \chi(t^m)$ for all $m \in \mathbb{Z}$. Since χ is continuous, we have $a_m = 0$ for all sufficiently large m. Thus it makes sense to define $k_0 \in \mathbb{Z}$ as the largest integer such that $a_{k_0} \neq 0$. We also set $K_0 = \max\{-1, k_0\}$ and set

$$U^A = t^{K_0+1}\mathbb{F}_p[\![t]\!].$$

Hence U^A is a compact open subgroup of A, and there exists a compact open subgroup U of G whose projection to $A \cong G/N$ coincides with U^A. Moreover, it follows from the definitions that $[U, U] \subseteq \ker(\chi)$.

Let $O_{\chi,U} = \{g \in G \mid [g, U] \subseteq \ker(\chi)\}$, as in Proposition 4.19. Let also $O^A_{\chi,U} \leqslant A$ be the image of $O_{\chi,U}$ under the canonical projection $G \to G/N \cong A$. An element $x = \sum_{i=i_0}^{\infty} x_i t^i \in A$ belongs to $O^A_{\chi,U}$ if and only if $\gamma(x, t^m) \in \ker(\chi)$ for all $m > K_0$. Using that χ is $\mathbb{F}_p$-linear, we deduce that

$$x \in O^A_{\chi,U} \Longleftrightarrow \sum_{i=i_0}^{\infty} \sigma_{i-m} a_{i+m} x_i = 0 \qquad \forall m > K_0. \tag{4.3.1}$$

Recall that $a_k = 0$ for all $k > k_0$. Thus, in the equations above, the terms indexed by any $i > k_0 - m$ are all zero, and we obtain

$$x \in O^A_{\chi,U} \Longleftrightarrow \sum_{i=i_0}^{k_0-m} \sigma_{i-m} a_{i+m} x_i = 0 \qquad \forall m > K_0. \tag{4.3.2}$$

We view that condition as a system of $\mathbb{F}_p$-linear equations that the coefficients $(x_i)_{i \geqslant i_0}$ of x must satisfy. Using again that $a_k = 0$ for all $k > k_0$, we see that all the equations corresponding to any $m > k_0 - i_0$ are trivial. Thus the system has only finitely many equations, one for each value of m satisfying $K_0 < m \leqslant k_0 - i_0$. We rewrite the system by numbering the equations by the number $\ell = k_0 - i_0 - m$

and by indexing the sum over $n = i - i_0$. We obtain $x \in O^A_{\chi,U}$ if and only if

$$(4.3.3)\quad \sum_{n=0}^{\ell} \sigma_{2i_0+\ell-k_0+n} a_{k_0-\ell+n} x_{i_0+n} = 0 \qquad \forall \ell \in \{0, 1, \dots, k_0 - i_0 - K_0 - 1\}.$$

The matrix defining that system is lower triangular. The diagonal entry on the ℓ^{th} row is

$$\sigma_{2i_0-k_0+2\ell} a_{k_0}.$$

Assume now that the condition (1) holds. Then the number of rows with a non-zero diagonal coefficient is at most c. In particular it is bounded independently of i_0. This implies that the $\mathbb{F}_p$-dimension of $O^A_{\chi,U}$ is finite. Therefore $O_{\chi,U}/N$ is finite. Since the group $L_{\chi,U}$ defined in Proposition 4.19 contains N, we infer that $O_{\chi,U}/L_{\chi,U}$ is finite. In view of Proposition 4.19 and Corollary 4.17, this confirms that G is type I.

Assume now that the condition (2) holds. We distinguish two cases.

Suppose first that $a_k = 0$ for all odd k. Observe that the integers $i - m$ and $i + m$ have the same parity. By assumption $\sigma_k = 0$ for all even k and $a_k = 0$ for all odd k. It follows that all the linear equations in the system (4.3.1) above are trivial in the case at hand. Hence $O^A_{\chi,U} = A$ so that $O_{\chi,U} = G$. Now an element $x \in A$ belongs to the projection of L_χ if and only if it satisfies the system (4.3.1) for all $m \in \mathbb{Z}$. Every equation in that system being trivial, we infer that $L_\chi = G$, hence $L_{\chi,U} = G$, so that the quotient $O_{\chi,U}/L_{\chi,U}$ is trivial in this case.

Suppose now that there exists an odd integer k such that $a_k \neq 0$. We let ℓ_0 be the largest such integer, so that $\ell_0 \leqslant k_0$. It follows that the first $k_0 - \ell_0$ rows of the matrix of the system (4.3.3) are trivial. After discarding them, we obtain a lower triangular matrix whose diagonal entries are of the form

$$\sigma_{2i_0+2\ell-2k_0+\ell_0} a_{\ell_0}.$$

As in case (1) above, we deduce from the hypotheses that the number of rows with a non-zero diagonal coefficient is bounded independently of i_0. This implies that the $\mathbb{F}_p$-dimension of $O^A_{\chi,U}$ is finite, so that $O_{\chi,U}/L_{\chi,U}$ is finite also in this case.

In particular $O_{\chi,U}/L_{\chi,U}$ is finite in both cases, hence G is type I by Proposition 4.19 and Corollary 4.17.

Assume finally that the condition (3) holds. For each $i \in \{0, 1, \dots, d-1\}$, we set $A_i = t^i \mathbb{F}_p(\!(t^d)\!) \leqslant A$ and define G_i as the preimage of A_i under the canonical projection $G \to G/N \cong A$. Each G_i is a closed normal subgroup of G, and we have $G = G_0 G_1 \dots G_{d-1}$. Moreover, in view of (3), we have $[G_i, G_j] = \{e\}$ for all $i \neq j$.

It follows that the product map

$$G_0 \times G_1 \times \cdots \times G_{d_1} \to G : (g_0, g_1, \ldots, g_{d-1}) \mapsto g_0 g_1 \ldots g_{d-1}$$

is a continuous surjective homomorphism. Recall that the type I property is preserved under forming direct products (see [**7**, Proposition 6.E.21(3)]). Therefore, to show that G is type I, it suffices to show that G_i is type I for each i.

Fix $i \in 0, 1, \ldots, d-1$. For all $a, b \in \mathbb{Z}$, we have $t^{i+ad}, t^{i+bd} \in A$ and

$$\gamma(t^{i+ad}, t^{i+bd}) = \sigma_{(a-b)d} t^{2i+(a+b)d}.$$

Set $N_2 = t^{2i}\mathbb{F}_p(\!(t^d)\!)$ and $N_1 = \bigoplus_{j \not\equiv 2i \mod d} t^j \mathbb{F}_p(\!(t^d)\!)$. Thus $\mathbb{F}_p(\!(t)\!) \cong N \cong N_1 \times N_2$, and we have $[G_i, G_i] \leqslant N_2$. In view of Corollary 4.18, we infer that G_i is type I if and only if G_i/N_1 is so.

We claim that G_i/N_1 has monomial commutation relations. In order to check this, we view N/N_1 as a subgroup of G_i/N_1. We identify the quotient $A_i \cong G_i/N \cong (G_i/N_1)/(N/N_1)$ with A via the isomorphism sending t^{i+nd} to t^n for all $n \in \mathbb{Z}$. We moreover identify the group $t^{2i}\mathbb{F}_p(\!(t^d)\!) = N_2 \cong N/N_1$ with A via the isomorphism sending t^{2i+nd} to t^n for all $n \in \mathbb{Z}$. After those identifications, we see that the commutator map of G_i satisfies the identity

$$\gamma(t^a, t^b) = \sigma_{(a-b)d} t^{a+b}$$

for all $a, b \in \mathbb{Z}$. Now the condition (3) implies that G_i satisfies the condition (1). Thus it is type I, and the proof is complete. □

Remark 4.34. Observe that in all cases covered by Theorem 4.33, the restriction of σ to $\mathbb{N}$ is an asymptotically periodic sequence, which is not asymptotically zero. That condition is however not sufficient for G to be type I, by virtue of the following result.

THEOREM 4.35. *Let G be a two-step nilpotent tdlc group with monomial commutation relations of type σ. Let $d > 0$ be an integer with $\sigma_d \neq 0$. If $\sigma_{dn} = 0$ for all $n \geqslant 2$, then G is not type* I.

PROOF. We shall prove that G has an open subgroup O admitting a discrete quotient that is not type I. The required conclusion follows (see Corollary 4.27).

Let $U^A = \mathbb{F}_p[\![t]\!]$ and U be a compact open subgroup of G whose image under the canonical projection $G \to G/N \cong A$ is U^A. Upon replacing U by a possibly larger group, we may assume that $U \cap N = t^{-n_0}\mathbb{F}_p[\![t]\!]$ for some $n_0 > 0$.

We also define

$$O^A = \left(\bigoplus_{n>0} \langle t^{-3nd} \rangle \oplus \langle t^{-3nd-d} \rangle \right) \oplus U^A,$$

and O be the preimage of O^A in G. Finally, we define

$$N_1 = \left(\bigoplus_{n>0, n \not\equiv -d \mod 6d} \langle t^{-n} \rangle \right) \oplus \mathbb{F}_p[\![t]\!],$$

viewed as a subgroup of $N \cong A$.

We claim that $[O, U] \subseteq N_1$. Since the commutator is constant on cosets of N in G, it suffices to show that for all integers $a, b \in \mathbb{Z}$ with $t^a \in O^A$ and $t^b \in U^A$, we have $\gamma(t^a, t^b) \in N_1$.

Let $t^a \in O^A$ and $t^b \in U^A$. We have $b \geqslant 0$. If $a \geqslant 0$, then the fact that $\gamma(t^a, t^b) \in N_1$ is clear. If $a < 0$, we have $a = -3nd$ or $a = -3nd - d$ for some $n > 0$. We infer that $\gamma(t^a, t^b) = -\sigma_{b+3nd} t^{b-3nd}$ or $-\sigma_{b+3nd+d} t^{b-3nd-d}$. If b is a multiple of d, then $\sigma_{b+3nd} = \sigma_{b+3nd+d} = 0$ by hypothesis, hence the required assertion follows. If b is not a multiple of d, then neither $b - 3nd$ nor $b - 3nd - d$ is congruent to $-d$ modulo $6d$. Thus we obtain that $\gamma(t^a, t^b) \in N_1$ in all cases.

The group N_1 is a central subgroup of O and $[O, U] \subseteq N_1$, it follows that UN_1 is an open normal subgroup of O. Let now

$$N_2 = \bigoplus_{n>n_0, n \equiv -d \mod 6d} \langle t^{-n} \rangle.$$

Hence $N_2 \cap UN_1 = \{e\}$. Finally, for each $n > n_0$, let g_n and $h_n \in O$ be preimages of t^{-3nd} and $t^{-3nd-d} \in O^A$ respectively. We have $[g_n, h_n] = \gamma(t^{-3nd}, t^{-3nd-d}) = \sigma_d t^{-d-6nd} \in N_2$. By hypothesis $\sigma_d \neq 0$, hence g_n and h_n do not commute. On the other hand, g_n commutes with g_m and h_m for all $m \neq n$.

Set $Q = O/UN_1$ and let $\pi \colon O \to Q$ be the canonical projection. For $n > n_0$, let Q_n be the subgroup of Q generated by $\pi(g_n)$ and $\pi(h_n)$. Since the restriction of π to N_2 is injective, it follows that the groups Q_n, which commute pairwise, are non-abelian. Moreover, the collection of their commutator groups $[Q_n, Q_n]$ taken over all $n > n_0$ generates their direct sum. In view of Corollary 4.26, we infer that Q is not virtually abelian, hence it is not type I (see Corollary 4.24). □

The following consequence of Theorem 4.35 is immediate. This recover the result [**27**, Theorem 5.9]

Corollary 4.36. *Let G be a two-step nilpotent tdlc group with monomial commutation relations of type σ. If σ is non-zero and finitely supported, then G is not type* I.

4.3.3. The Glöckner–Willis contraction groups. We now apply our results to a class of two-step nilpotent groups first defined by Glöckner–Willis in [**59**, Section 8]. The study of the unitary representations of those groups was initiated by the second author in [**27**]. Our goal is to strengthen and extend the results from loc. cit.

As before, we let p be a prime and set $A = \mathbb{F}_p(\!(t)\!)$. Let $\nu\colon A \to \mathbb{Z}$ be the standard valuation.

Definition 4.37. Given a sequence $s\colon \mathbb{N}_{>0} \to \{0,1\}$, we define the map

$$\eta_s\colon A \times A \to A$$

on elements $x := \sum_{i=\nu(x)}^{\infty} x_i t^i, y := \sum_{i=\nu(y)}^{\infty} y_i t^i \in \mathbb{F}_p(\!(t)\!)$ by

$$\eta_s(x,y) := \sum_{k\in\mathrm{supp}(s)} \sum_{i=\nu(x)}^{\infty} x_i y_{i+2k} t^{i+k}.$$

The following is shown in [**59**].

Proposition 4.38 (See [**59**, Section 8])**.** *For any $s \in \{0,1\}^{\mathbb{N}_{>0}}$, the map η_s is a continuous bi-additive 2-cocycle on A, and it is equivariant in the sense that $\eta_s(tx,ty) = t\eta_s(x,y)$ for all $x,y \in A$. Furthermore, the corresponding central extension, denoted $G = A \times_{\eta_s} A$, is a contraction group with respect to the automorphism α that is multiplication by t on each factor. For distinct sequences $s \in \{0,1\}^{\mathbb{N}_{>0}}$, the corresponding groups are not isomorphic.*

Since η_s is continuous and bi-additive, it is determined by its values on pairs (t^a, t^b). We observe that

$$\eta_s(t^a,t^b) = \begin{cases} s(k)t^{a+k} & \text{if } b = a+2k \text{ for some } k > 0,\\ 0 & \text{otherwise} \end{cases}$$

for all $a, b \in \mathbb{Z}$.

The group $G = A \times_{\eta_s} A$ is defined as the set $A \times A$ endowed with the multiplication defined by

$$(x,a).(y,b) = (x+y, a+b+\eta_s(x,y)).$$

In particular, the second factor $N = \{(0,a) \mid a \in A\}$ in G corresponds to the canonical central subgroup isomorphic to A, and $G/N \cong A$. Viewing those isomorphisms as identification, we may consider the commutator map $\gamma_s\colon A \times A \to A$ associated with G as in Section 4.2.3. One can check that the following identities hold for all $a, b \in \mathbb{Z}$:

$$\gamma_s(t^a,t^b) = \begin{cases} 0 & \text{if } a = b \text{ or } a \not\equiv b \mod 2,\\ s(\frac{b-a}{2})t^{\frac{a+b}{2}} & \text{if } a < b \text{ and } a \equiv b \mod 2,\\ -s(\frac{a-b}{2})t^{\frac{a+b}{2}} & \text{if } a > b \text{ and } a \equiv b \mod 2. \end{cases} \tag{4.3.4}$$

For $i = 0, 1$, set $A_i = t^i\mathbb{F}_p(\!(t^2)\!)$. We have $A \cong A_0 \oplus A_1$. Let G_i be the preimage of A_i under the canonical projection $G \to G/N \cong A$.

Proposition 4.39. *For any sequence* $s \in \{0,1\}^{\mathbb{N}_{>0}}$, *the group* $G = A \times_{\eta_s} A$ *has the following properties.*

- *(i)* G_0 *and* G_1 *are closed normal subgroups of* G *that commute. Moreover, we have* $G = G_0 G_1$.
- *(ii)* G_0 *is isomorphic to* G_1.
- *(iii)* G *is type* I *if and only if* G_0 *is type* I.
- *(iv)* G_0 *is isomorphic to a two-step nilpotent tdlc group with monomial commutation relations of type* σ, *where* $\sigma \colon \mathbb{Z} \to \mathbb{F}_p$ *is the sequence defined by*

$$\sigma_z = \begin{cases} s(z) & \text{if } z > 0, \\ 0 & \text{if } z = 0, \\ -s(-z) & \text{if } z < 0. \end{cases}$$

Proof. The commutation relations (4.3.4) imply that G_0 and G_1 commute. Hence the assertion (i) follows readily from the definitions.

By Proposition 4.38, the group G has a contractive automorphism α that acts as the multiplication by t on $N \cong A$ and $G/N \cong A$. It follows that the restriction of α to G_0 is an isomorphism of G_0 onto G_1. Thus (ii) holds.

Suppose that G_0 is type I, so that G_1 is also type I by (ii). Hence the direct product $G_0 \times G_1$ is type I by [**7**, Proposition 6.E.21(3)]. By (i), the multiplication map yields a surjective continuous homomorphism $G_0 \times G_1 \to G$. Hence G is type I as well.

Assume conversely that G_0 is not type I. In particular it is non-abelian, hence s is non-zero. It follows from [**59**, Lemma 8.4] that N is the center of G and of G_0. By Corollary 4.17, there exists a character $\chi \in \widehat{N}$ such that $\omega_\chi \colon G_0/N \to \widehat{G_0/N}$ has a non-closed image. Let $U \leqslant G$ be a compact open subgroup of G such that $[U, U] \leqslant \ker(\chi)$, and set $U_0 = U \cap G_0$ and $U_1 = U \cap G_1$. Since G is second countable, the surjective homomorphism $G_0 \times G_1 \to G$ is an open map, hence $U_0 U_1$ is open in G. Therefore, upon replacing U by the smaller subgroup $U_0 U_1$, we may assume that $U = U_0 U_1$.

Let $O_{\chi, U_0} \leqslant G_0$ be the group defined in Proposition 4.19. The latter implies that the quotient $O_{\chi, U_0}/\ker(\chi)$ is not center-by-finite. Since $U = U_0 U_1$ and since G_0 commutes with G_1, we infer that $O_{\chi, U_0} \leqslant O_{\chi, U}$. Hence $O_{\chi, U}/\ker(\chi)$ is not center-by-finite. Using Proposition 4.19 again, we deduce that $\omega_\chi \colon G/N \to \widehat{G/N}$ has a non-closed image. Thus G is not type I by Corollary 4.17. This proves that (iii) holds.

To prove (iv), we observe that G_0 contains N, and that $G_0/N \cong A_0 = \mathbb{F}_p(\!(t^2)\!)$. The commutator map of G_0 is the map $\gamma_s \colon A_0 \times A_0 \to N$ induced by the commutator relations appearing in (4.3.4). Consider the isomorphism $\delta \colon A \to A_0$ sending t^n to t^{2n} for all $n \in \mathbb{Z}$, and the isomorphism $\mu \colon A \to N$ sending x to $-x$ for all x. Define

$\gamma = \mu \circ \gamma_s \circ (\delta \times \delta)\colon A \times A \to A$. We deduce from (4.3.4) that

$$\gamma(t^a, t^b) = \sigma_{a-b} t^{a+b}$$

for all $a, b \in \mathbb{Z}$, where σ is the bi-infinite sequence defined in the statement of the proposition. The assertion (iv) readily follows. $\square$

Theorem 4.8 stated in the introduction is now a direct consequence of Theorems 4.33 and 4.35.

4.4. Type I extensions

The goal of this section is to complete the proof of Theorem 4.11 stated in the introduction.

Proof of Theorem 4.11. Set $A = G/N$. Since A is abelian, the desired conclusion is clear if χ is the trivial character. We assume henceforth that χ is non-trivial.

Let $Q = G/\ker(\chi)$ and denote the canonical projection by $g \mapsto \bar{g}$. Since $\ker(\chi) \leqslant N$, the quotient map $G \to G/N = A$ factors through Q. In particular, every character $\varphi \in \widehat{A}$ may be viewed as a character of Q (and also of G).

Since N is of exponent p, it follows that $C = N/\ker(\chi) \cong \mathrm{im}(\chi)$ is a cyclic group of order p. Similarly, the image of each $\varphi \in \widehat{A}$ is contained in the cyclic subgroup of order p of $\mathbb{C}^*$, which coincides with $\mathrm{im}(\chi)$. Upon identifying C with $\mathrm{im}(\chi)$ via χ, we infer that the dual $\widehat{A}$ acts on the quotient group $Q = G/\ker(\chi)$ via

$$\widehat{A} \times Q \to Q : (\varphi, \bar{g}) \mapsto \varphi(\bar{g})\bar{g}.$$

One checks that this is a continuous action by automorphisms on Q. Therefore, we may form the semi-direct product

$$E = \widehat{A} \ltimes Q.$$

By construction, the group Q embeds as a closed normal subgroup of E. The group E consists of ordered pairs $(\varphi, \bar{g}) \in \widehat{A} \times Q$ with multiplication defined by

$$(\varphi, \bar{g}) \cdot (\psi, \bar{h}) = (\varphi\psi, \psi(\bar{g})^{-1}\bar{g}\bar{h}).$$

Let $\bar{C} = \{(1, \bar{c}) \mid \bar{c} \in C\}$.

We claim that E is a two-step nilpotent locally compact group. More precisely, we claim that $[E, E] \leqslant \bar{C} \leqslant Z$, where Z denotes the center of E. Indeed, one computes that for all $(\varphi, \bar{g}), (\psi, \bar{h}) \in E$, we have

$$[(\varphi, \bar{g}), (\psi, \bar{h})] = \big(1, [\bar{g}, \bar{h}]\psi(\bar{g})^{-1}\varphi(\bar{h})\big) \in \bar{C}.$$

Thus we have $[E,E] \leqslant \bar{C}$. Let us now assume that $\bar{g} \in C \leqslant Q$. It then follows that $[(1,\bar{g}),(\psi,\bar{h})] = \big(1,[\bar{g},\bar{h}]\psi(\bar{g})^{-1}\big) = e$, since the elements of C are central in Q, and since every $\psi \in \widehat{A}$, viewed as a character of Q, is trivial on C. The claim is proved.

We next claim that E is type I. To this end, we rely on Corollary 4.17. Since $[E,E] \leqslant \bar{C} \leqslant Z$ by the previous claim, and since the map $\bar{c} \mapsto (1,\bar{c})$ is an isomorphism of C onto $\bar{C}$, it suffices to show that for every $\sigma \in \widehat{C}$, the homomorphism

$$\omega_\sigma \colon E \to \widehat{E/\bar{C}} : (\varphi,\bar{g}) \mapsto \omega_\sigma(\varphi,\bar{g}) \colon \begin{array}{ccc} E/\bar{C} & \to & \mathbb{C}^* \\ (\psi,\bar{h})\bar{C} & \mapsto & \sigma\big([\bar{g},\bar{h}]\psi(\bar{g})^{-1}\varphi(\bar{h})\big) \end{array}$$

has a closed image. This is clear if σ is trivial. We finish the proof of the claim by showing that if σ is non-trivial, the homomorphism ω_σ is surjective.

We have $E/\bar{C} \cong \widehat{A} \times A$, so that $\widehat{E/\bar{C}} \cong A \times \widehat{A}$. Moreover A is of exponent p, so its dual $\widehat{A}$ is also of exponent p. Taking $\bar{g} = 1$, we see that $\omega_\sigma(\varphi,1) \colon (\psi,\bar{h})\bar{C} \mapsto \sigma \circ \varphi(\bar{h})$. Since σ is injective, we have $\{\sigma \circ \varphi \mid \varphi \in \widehat{A}\} = \widehat{A}$, hence we obtain $\{\omega_\sigma(\varphi,1) \mid \varphi \in \widehat{A}\} = \{1\} \times \widehat{A}$. Given $\bar{g} \in Q$, we now set $\varphi_{\bar{g}} \colon \bar{h} \mapsto [\bar{g},\bar{h}]^{-1}$. Since C is identified with $\mathrm{im}\chi$, we may view $\varphi_{\bar{g}}$ as a character of A. It follows that for each $\bar{g} \in Q$, we have $\omega_\sigma(\varphi_{\bar{g}},\bar{g}) \colon (\psi,\bar{h})\bar{C} \mapsto \sigma \circ \psi(\bar{g})^{-1}$. Using again that σ is injective, we obtain $\{\omega_\sigma(\varphi_{\bar{g}},\bar{g}) \mid \bar{g} \in Q\} = \widehat{(\widehat{A})} \times \{1\} \cong A \times \{1\}$. It follows that the image of ω_σ contains the full direct product $\widehat{E/\bar{C}} \cong A \times \widehat{A}$, thereby confirming that ω_σ is surjective.

We have shown that Q continuously embeds as a closed normal subgroup of E, which is a two-step nilpotent second countable locally compact group of type I. We henceforth view Q as a closed normal subgroup of E. Since A and N have exponent p, they are both totally disconnected, hence G, Q and E are all totally disconnected as well. Let $U \leqslant E$ be a compact open subgroup. Then QU is an open subgroup of E, hence it is type I by [**7**, Proposition 6.E.21(1)]. Thus Q is a closed cocompact normal subgroup of QU, which is a two-step nilpotent type I group. ☐

Remark 4.40. Every non-type I contraction group G afforded by Theorem 4.8 has a factor representation that is not type I. Letting χ be the central character of such a factor representation, we infer that the quotient $G/\ker(\chi)$ is not type I. Theorem 4.11 applies to G, hence we obtain numerous non-type I groups admitting an extension by a compact group that is type I.

CHAPTER 5

The affine group of a local field is Hermitian

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Abstract

The question of whether the group $\mathbb{Q}_p \rtimes \mathbb{Q}_p^*$ is Hermitian has been stated as an open question in multiple sources in the literature, even as recently as a paper by R. Palma published in 2015. In this note we confirm that this group is Hermitian by proving the following more general theorem: given any local field $\mathbb{K}$, the affine group $\mathbb{K} \rtimes \mathbb{K}^*$ is a Hermitian group. The proof is a consequence of results about Hermitian Banach $*$-algebras from the 1970's. In the case that $\mathbb{K}$ is a non-archimedean local field, this result produces examples of totally disconnected locally compact Hermitian groups with exponential growth, and these are the first examples of groups satisfying these properties. This answers a second question of Palma about the existence of such groups.

5.1. Introduction

A locally compact group G, or equivalently the Banach $*$-algebra $L^1(G)$, is called *Hermitian* if all self-adjoint $f \in L^1(G)$ have real spectrum. It is a classical question in harmonic analysis and Banach algebra theory to determine which locally compact groups are Hermitian [**88**]. This question has been studied extensively since the mid-1900's, particularly in the cases of connected Lie groups and discrete groups. See [**115**, Section 12.6.22] and [**131**] for more details on the current state of knowledge.

It is well known that the real $ax + b$ group, $\mathbb{R} \rtimes \mathbb{R}_{>0}$, is a Hermitian group. This fact is typically attributed to the following German paper of Leptin [**85**, Satz 6], but also follows from a later classification of solvable connected simply-connected Hermitian Lie groups of dimension $\leqslant 6$ [**89, 118**]. On the other hand, it is stated as an open question in, for example, [**115**, Page 1490] and [**113**, Section 3.6], to determine whether the group $\mathbb{Q}_p \rtimes \mathbb{Q}_p^*$ is Hermitian. In this note, we answer this question by proving the following more general theorem.

Theorem A. Let $\mathbb{K}$ be a local field. The group $\mathbb{K} \rtimes \mathbb{K}^*$ is Hermitian.

When $\mathbb{K}$ is a non-achimedean local field, the group $\mathbb{K} \rtimes \mathbb{K}^*$ is a non-discrete totally disconnected locally compact group. Furthermore, this group is non-unimodular and hence has exponential growth. This theorem thus produces examples of (non-discrete) totally disconnected locally compact Hermitian groups with exponential growth, answering a question of Palma about the existence of such groups, see [**112**, Question 2, page 266] and [**113**, Section 3.6]. In contrast, a finitely generated solvable discrete group is Hermitian if and only if it has polynomial growth [**86**, Page 277]. It should also be noted that, as far as the author is aware, it is still an open question as to whether there exists a discrete Hermitian group with exponential growth, see [**112, 113**] again.

The proof of Theorem A is a consequence of Satz 5 in Leptin's original article [**85**] (we will refer to this theorem as *Leptin's theorem* from now on). The point of this note is to translate the critical parts of Leptin's paper [**85**] into English, and provide the few extra details required to verify that $\mathbb{K} \rtimes \mathbb{K}^*$ is a Hermitian group whenever $\mathbb{K}$ is a local field.

If we now instead take $\mathbb{K}$ to be a number field equipped with the discrete topology, then it is also well known that the discrete group $\mathbb{K} \rtimes \mathbb{K}^*$ is *not* a Hermitian group since it contains free subsemigroups on two or more generators [**72**]. To conclude the article, we discuss this and related examples, and show that the assumptions of Leptin's theorem are not satisfies in these cases. This has connections with other ongoing work of the author concerning the Hermitian property for semi-direct products.

5.2. Preliminaries on generalised L^1-algebras and Fourier algebras

In this section we define some notation and terminology concerning generalised L^1-algebras and Fourier algebras. We refer the reader to [**83, 84, 86**] for more details on generalised L^1-algebras. Throughout this section G is a locally compact group and all integration is performed against a fixed left Haar measure on G.

Let A be a Banach $*$-algebra and G a locally compact group acting strongly continuously on A by isometric $*$-automorphisms. Given $a \in A$ and $x \in G$, let a^x denoted the image of a under x with respect to this action. The *generalised L^1-algebra* associated to G and A is the space $L^1(G, A)$ of measurable A-valued functions on G which are integrable against the Haar measure. It forms a Banach $*$-algebra with the product

$$f \star g(x) := \int_G f(xy)^{y^{-1}} g(y^{-1})\, dy$$

and involution

$$f^*(x) := \Delta(x)^{-1}(f(x^{-1})^x)^*$$

where Δ is the modular function on G. The norm on $L^1(G, A)$ is the obvious one:

$$\|f\|_{L^1(G,A)} := \int_G \|f(x)\|_A \, dx.$$

Now let's further assume that G is abelian. Then, the *Fourier algebra* of G, denoted $A(G)$, is the image of $L^1(\widehat{G})$ under the Fourier transform on $\widehat{G}$. The algebra $A(G)$ is a $*$-subalgebra of $C_0(G)$, and it is a Banach $*$-algebra when equipped with a certain norm which we denote by $\|\cdot\|_{A(G)}$. We will not go further into the details of Fourier algebras here, but the reader can consult the book [**74**] for more details if desired.

We now note the following result which will be critical in this article.

Proposition 5.1. *Let $G := N \rtimes H$ be a semi-direct product of locally compact abelian groups with N normal in G. Then, we have isometric $*$-isomorphisms:*

$$L^1(G) \cong L^1(H, L^1(N)) \cong L^1(H, A(\widehat{N})).$$

PROOF. The first isomorphism follows from standard results on generalised L^1-algebras, see [**83**, Satz 10]. The second isomorphism follows from the fact that $L^1(N)$ is isometrically $*$-isomorphic to $A(\widehat{N})$ for any locally compact abelian group N, see [**74**, Remark 2.4.5]. □

5.3. Translations from Leptin's article

In this section we translate the necessary parts of Leptin's article [**85**] to English and set some further notation.

Let G be a locally compact group and X a locally compact G-space. Let $A \subseteq C_0(X)$ be a $*$-subalgebra satisfying the following four properties:

(i) A is a Banach $*$-algebra with respect to some norm denoted $\|f\|_A$ (possibly different to the uniform norm on $C_0(X)$);
(ii) A is left translation invariant i.e. for $f \in A$ and $g \in G$, $f^g(x) := f(gx)$ is in A and $\|f^g\|_A = \|f\|_A$;
(iii) For every $f \in A$, the map $g \mapsto f^g$ is continuous from G to A;
(iv) A is a regular Banach algebra and the functions of compact support in A are dense in A.

Now, for a closed subset $T \subseteq X$, define

$$k(T) := \{f \in A : f(x) = 0 \; \forall x \in T\}$$

and

$$j(T) := \overline{\{f \in A \cap C_c(X) : \operatorname{supp}(f) \cap T = \varnothing\}}.$$

The sets $k(T)$ and $j(T)$ are respectively the largest and smallest ideals of A with zero set T, in particular, $j(T) \subseteq k(T)$. The set T is called a *set of synthesis* or *Wiener* (with respect to A) if $j(T) = k(T)$. Then, Leptin proves the following theorem in [**85**], which we refer to as *Leptin's theorem.*

Theorem 5.2. [**85**, Satz 5] *Suppose that G is a σ-compact locally compact abelian group and X a locally compact G-space. Let A be a $*$-subalgebra of $C_0(X)$ satisfying properties (i) through to (iv) as above. If the action of G on X has finitely many orbits, and the closed G-invariant subsets of X are sets of synthesis, then the Banach $*$-algebra $L^1(G, A)$ is Hermitian.*

The following result is then a direct corollary of the theorem. It is not stated explicitly in Leptin's article [**85**], but it is alluded to in the discussion of the article.

Corollary 5.3. *Let $G := N \rtimes H$ be a semi-direct product of locally compact abelian groups with N normal in G and H σ-compact. If the action $H \curvearrowright \widehat{N}$ has finitely many orbits, and the closed H-invariant subsets of $\widehat{N}$ are sets of synthesis with respect to $A(\widehat{N})$, then G is Hermitian.*

Proof. By Proposition 5.1, we have an isomorphism $L^1(G) \cong L^1(H, A(\widehat{N}))$, so it suffices to check that $L^1(H, A(\widehat{N}))$ is Hermitian. By standard facts on Fourier algebras (see [**74**, Section 2]), the Fourier algebra $A(\widehat{N})$ satisfies properties (i) through to (iv) as listed at the start of this section, where we take $X = \widehat{N}$. The corollary then follows by Theorem 5.2. □

5.4. Proof of Theorem A

We now prove Theorem A from the introduction. Throughout this section we assume that $\mathbb{K}$ is a local field. To prove Theorem A, we just need to show that the action $\mathbb{K}^* \curvearrowright \widehat{\mathbb{K}}$ satisfies the assumptions as in Corollary 5.3, where $H = \mathbb{K}^*$ and $N = \mathbb{K}$.

To do this, we first remind the reader of the following description of the Pontryagin dual $\widehat{\mathbb{K}}$.

Lemma 5.4. [**18**, Proposition 19, page 234] *Let $\mathbb{K}$ be a local field, viewed as an abelian group under addition, and let χ be a non-trivial unitary character of $\mathbb{K}$. Given $y \in \mathbb{K}$, define $\chi_y(x) := \chi(yx)$ for $x \in \mathbb{K}$. Then, the map*

$$\mathbb{K} \to \widehat{\mathbb{K}}, y \mapsto \chi_y$$

is an isomorphism of locally compact abelian groups.

One then uses this lemma to compute that for $x, y \in \mathbb{K}$ and $z \in \mathbb{K}^*$,

$$z \cdot \chi_y(x) = \chi_y(zx) = \chi(yzx) = \chi_{zy}(x).$$

In particular, it follows that under the identification $\mathbb{K} \cong \widehat{\mathbb{K}}$ as in Lemma 5.4, the action of $\mathbb{K}^*$ on $\widehat{\mathbb{K}}$ is identified with the canonical multiplicative action of $\mathbb{K}^*$ on $\mathbb{K}$. Thus, we will work with the action $\mathbb{K}^* \curvearrowright \mathbb{K}$ instead for the remainder of the proof.

Then, one computes easily that there are two orbits of the action $\mathbb{K}^* \curvearrowright \mathbb{K}$: $\{0\}$ and $\mathbb{K}\backslash\{0\}$. Furthermore, the closed $\mathbb{K}^*$-invariant subsets of $\mathbb{K}$ are $\{0\}$ and $\mathbb{K}$. To complete the proof, we just need to show that the sets $\{0\}$ and $\mathbb{K}$ are sets of synthesis with respect to the algebra $A(\mathbb{K})$.

Clearly $j(\mathbb{K}) = k(\mathbb{K})$ since both $j(\mathbb{K})$ and $k(\mathbb{K})$ contain only the zero function. To complete the proof we just need to show that $j(\{0\}) \supseteq k(\{0\})$ (and hence $j(\{0\}) = k(\{0\})$). Suppose that $f \in k(\{0\}) \subseteq A(\mathbb{K})$. We must show that $f \in j(\{0\})$. For each $n \in \mathbb{N}$, by [**74**, Lemma 2.3.7], there exists a function $h_n \in A(\mathbb{K}) \cap C_c(G)$ satisfying the following:

(i) h_n vanishes in a neighbourhood of $0 \in \mathbb{K}$;
(ii) $\|h_n - f\|_{A(G)} \leqslant 1/n$.

In particular, it follows that $h_n \in j(\{0\})$ for each $n \in \mathbb{N}$ and h_n converges to f in $A(\mathbb{K})$ as $n \to \infty$. Thus $f \in j(\{0\})$ by definition of $j(\{0\})$. This completes the proof.

5.5. Remarks

(i) Let $\mathbb{K}$ be a number field, in particular, this field is countable and has the discrete topology. As mentioned in the introduction, since $\mathbb{K} \rtimes \mathbb{K}^*$ is discrete and contains free subsemigroups on two or more generators, it follows by the results of [**72**] that this semi-direct product is not a Hermitian group. Let $\mathbb{A}_{\mathbb{K}}$ denote the ring of adeles over the number field $\mathbb{K}$. It is well known that we have an isomorphism $\widehat{\mathbb{K}} \cong \mathbb{A}_{\mathbb{K}}/\mathbb{K}$ [**126**, Proposition 7.15]. Furthermore, $\widehat{\mathbb{K}}$ is uncountable since it is a quotient of an uncountable group by a countable group. Since $\mathbb{K}^*$ is countable, the orbits of the action $\mathbb{K}^* \curvearrowright \widehat{\mathbb{K}}$ are countable, hence there must be uncountably many orbits for this action since $\widehat{\mathbb{K}} \cong \mathbb{A}_{\mathbb{K}}/\mathbb{K}$ is uncountable. Thus one of the assumptions in Leptin's theorem does not hold in this case. It is not clear to the author whether the closed $\mathbb{K}^*$-invariant subsets of $\widehat{\mathbb{K}}$ are sets of synthesis.

(ii) Let $\mathbb{Z}$ act on $\mathbb{Q}_p$ by multiplication by p and form the corresponding semi-direct product $\mathbb{Q}_p \rtimes \mathbb{Z}$. It is posed as an open question in [**113**, Section 3.6] and a recent article by the author [**29**, Section 8.5], to determine whether the group $\mathbb{Q}_p \rtimes \mathbb{Z}$, and certain generalisations of this group, are Hermitian.

By a similar argument to the last section, it can be checked that the induced action of $\mathbb{Z}$ on $\widehat{\mathbb{Q}_p}$ has infinitely many orbits. The same is the case for the various generalisations of this group stated in [**113**] and [**29**]. In particular, the example of the last section shows that we cannot expect to be able to generalise Corollary 5.3 to prove Hermitianness of the group $\mathbb{Q}_p \rtimes \mathbb{Z}$ and its generalisations. New results will need to be used/developed to understand the Hermitian property of these semi-direct products.

If these groups are proved to be Hermitian, then this will provide further examples of non-discrete totally disconnected locally compact Hermitian groups with exponential growth.

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