On Two Properties of Totally Disconnected Locally Compact Groups

Max Carter

under the supervision of

George Willis

Mathematics Discipline, School of Mathematics and Physical Sciences, University of Newcastle

Thesis submitted for the degree of

Bachelor of Mathematics (Honours)

November 2020

Abstract

In previous work it was shown that the automorphism group of a label-regular tree, denoted $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$, can be decomposed into a Cartan-like decomposition, moreover, the coset representatives in the decomposition satisfy the *contraction group property*: every unbounded sequence of coset representatives has a subsequence with non-trivial contraction group. This leads to the proof that the range of every continuous homomorphism from the simple subgroup of $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$ generated by edge stabilisers is closed, and we say that this subgroup has the *closed range property.*

In the present article, after giving the reader an introduction to totally disconnected locally compact groups acting on trees and buildings, we study these contraction group and closed range properties in a larger class of totally disconnected locally compact groups, resulting in closedness of range results for a variety of different simple totally disconnected locally compact groups. We also study the contraction group and closed range properties in more generality, in particular, we answer the question of whether the contraction group property depends on the choice of compact open subgroup or choice of coset representatives in our Cartan-like decomposition, and whether the closed range property passes to subgroups and supergroups.

Contents

CHAPTER 1

Introduction

Broadly speaking, group theory is the mathematical study of symmetries through the study of an algebraic structure called a group. A common question in group theory research over the past century or more has been concerned with the classification, or the attempt to classify, large classes of groups. Currently there is a classification for finite simple groups, however, much less is known about the class of infinite groups. Locally compact topological groups are a natural class of groups, containing all finite groups and many infinite groups, and they appear in numerous applications across all of mathematics. Modern research is concerned with classifying and building a structure theory for the class of locally compact topological groups. Every locally compact group G admits a short exact sequence:

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow G/G_0 \longrightarrow 1$$

where G_0 denotes the connected component of the identity in G, which forms a closed connected locally compact normal subgroup of G. Hence, understanding any locally compact group G essentially reduces to understanding the connected locally compact subgroup G_0, and the totally disconnected locally compact quotient G/G_0.

Connected locally compact groups are already fairly well understood: in work by Gleason, Montgomery and Zippin [**Gle51, Gle52, MZ52**] to solve Hilbert's fifth problem, connected locally compact groups have been identified as inverse limits of connected Lie groups. Thus the well developed techniques of Lie theory can be used to understand the class of connected locally compact groups. Totally disconnected locally compact groups (t.d.l.c. groups from now on) on the other hand are not as well understood, and for many years the only known general result for t.d.l.c. groups was a theorem by van Dantzig from 1931 (c.f. [**vD31, vD36**]), which asserts that every t.d.l.c. group admits a basis of compact open subgroups. It wasn't until the 90's, when Willis published the paper 'Structure Theory of Totally Disconnected Locally Compact Groups' [**Wil94**] that significant advances started to be made in understanding t.d.l.c. groups. In this paper, Willis studies the space of compact open subgroups of a t.d.l.c. group, and introduces the notion of the scale function and tidy subgroups for t.d.l.c. groups, which allow for arguments of dynamical nature to be made and has formed a significant contribution to the structure theory of t.d.l.c. groups. As a result, rapid progress is now being made in constructing a structure theory for t.d.l.c. groups, however, much more work is still needed.

Some of the most recent progress in the study of t.d.l.c. groups has been in understanding the class of compactly generated t.d.l.c. groups. Indeed, every t.d.l.c. group can be recognised as a directed union of compactly generated open subgroups, hence, understanding the compactly generated ones can assist in understanding the broader picture. Recent work for instance by Caprace–Monod in [**CM11**], Caprace–De Medts in [**CDM11**] and Caprace–Reid–Willis in [**CRW17a, CRW17b**] have all contributed significant advances to the study of compactly generated t.d.l.c. groups, and a better picture of these groups is now presenting itself.

The Cayley-Abels graph construction illustrates the ease of working with compactly generated t.d.l.c. groups and the significance of automorphism groups of graphs in the theory of t.d.l.c. groups: the Cayley-Abels graph associated to a compactly generated t.d.l.c. group G, is a locally finite connected graph that G acts on vertex-transitively with compact open vertex stabilisers, and generalises the idea of a Cayley graph of a finitely generated group to compactly generated t.d.l.c. groups (see [**KM08**] for more details). It is a known result that every compactly generated t.d.l.c. group can be represented as a group of symmetries of its corresponding Cayley-Abels graph. It is also true that the automorphism group of every connected locally finite graph is a t.d.l.c. group, hence, the study of compactly generated t.d.l.c. groups is more or less coextensive with the study of automorphisms of connected locally finite graphs. As a result, a prominent feature of the study of t.d.l.c. groups over the past couple of decades has been in understanding automorphism groups of locally finite connected graphs, in particular, infinite locally finite trees, and this has turned out to be an extensive and fruitful area of study. Those groups with a non-discrete topology are of utmost importance to the structure theory of t.d.l.c. groups.

In the present article, after introducing some basic concepts and notation in Chapter 2, we begin in Chapter 3 by surveying the current literature on automorphism groups of locally finite trees. This includes the universal groups construction seen in [**BM00**] and some of its generalisations, as well as the k-closure construction from [**BEW15**]. These groups all provide a large array of examples of (non-discrete) compactly generated t.d.l.c. groups acting on trees as discussed in the previous paragraph. Neretin's groups, or almost automorphism groups of trees, are also surveyed in this chapter. These are groups acting on the boundary of a rooted tree and provide further examples of non-discrete compactly generated t.d.l.c. groups. This chapter provides a strong foundation for Chapter 4 and Chapter 5 where we build upon the results seen in the paper [**CW20**].

The article [**CW20**] follows a recent trend in work that involves taking ideas from the theory of Lie groups and algebraic groups, and testing whether similar results hold for t.d.l.c. groups. In this paper, a certain type of infinite labelled tree called a label-regular tree is studied, and it was shown that the automorphism groups of these trees admit Bruhat and Cartan type decompositions as typically seen in the theory of Lie groups and algebraic groups. As a result of these decompositions, it is also shown that continuous homomorphisms from the simple subgroup generated by edge stabilisers have closed range, analogous to a result for simple Lie groups. This is a result of a more general argument: simple groups admitting a Cartan-like decomposition satisfying a property, called the *contraction group property*, also satisfy the property that every continuous homomorphism from the group has closed range, which we call the *closed range property*. In Chapter 4, we study the contraction group and closed range properties in greater detail than what was seen in [**CW20**]. In particular, we determine whether the contraction group property depends on choice of compact open subgroup or coset representatives in our Cartan-like decomposition. We also show that the closed range property passes to supergroups under certain circumstances.

In Chapter 5 we investigate the contraction group and closed range properties in a broader class of automorphism groups of trees that we introduced in Chapter 3. In particular, we look at the groups $G(F, F')$ studied by Le Boudec in [**LB16**]. These groups do not satisfy the closed range property in most cases since they are not closed in $\mathrm{Aut}(\mathcal{T}_d)$ (except in trivial cases), and as a result, it is expected that they will provide examples of groups that do not satisfy the contraction group property. We provide an example of a decomposition of one of these groups to illustrate how the contraction group property can fail. We then prove some more general closed range results for groups satisfying a generalised version of Tits' independence property called Property P_k. The main result is the following theorem:

Theorem 5.4. Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ be a closed subgroup and suppose that G does not stabilise any proper non-empty subtree, or fix an end of $\mathcal{T}$. If G satisfies Property P_k, then G^{+_k} has the closed range property.

Some nice corollaries concerning (generalised) universal groups and groups acting on trees with semiprimitive locally action stem from this result.

Another combinatorial structure that is of interest to us, and is more general than trees, is the notion of a building. Buildings are simplicial complexes with certain symmetry properties and were originally introduced by Jacque Tits as a means of classifying certain algebraic groups. The automorphism groups of certain types of buildings, such as semi-regular right-angled buildings, form another class of compactly generated t.d.l.c. groups (provided that the building is locally finite) and hence are another interesting class of groups to study in the theory of t.d.l.c. groups. In Chapter 6 we give a brief overview of some recent work on automorphism group of semi-regular right-angled buildings as well as a generalisation of the Universal groups construction for these buildings. The article concludes in Chapter 7 where we initiate the study of the contraction group and closed range properties for automorphism groups of buildings.

CHAPTER 2

Preliminaries

In this chapter we give a brief overview of the graph theory and group theory knowledge required in this article. This also gives us a chance to lay out the notation that will be used in the sequel.

2.1. Graphs and Group Actions

By a *graph*, we mean a pair $\Gamma = (V\Gamma, E\Gamma)$, where $V\Gamma$ is the collection of vertices, and $E\Gamma$ is the collection of edges of the graph Γ. The edges are unordered pairs of vertices from $V\Gamma$ i.e. we will be considering undirected graphs. We also assume that our graphs are *simple*, meaning they do not contain loops or double edges, and they are connected. A *tree* is a connected graph with no cycles. A vertex of a graph is said to be a *leaf* if it has valency 1, that is, has only 1 edge connected to it, and a graph is called *locally finite* if every vertex has only finitely many edges connected to it. The *regular tree* of valency d, denoted $\mathcal{T}_d$, is the infinite tree with the property that every vertex has d adjacent vertices, where two vertices are *adjacent* if they have an edge connecting them.

A *path* in an infinite graph Γ is a sequence of vertices $(v_i)_{i \in I} \subseteq V\Gamma$, where I is some (at most countable) indexing set, v_i is adjacent v_{i+1}, and $v_i \neq v_{i+2}$ for all $i \in I$. We call a path a *ray* if $I = \mathbb{N}$ and a *bi-infinite path* if $I = \mathbb{Z}$. An *end* of an infinite graph is an equivalence class of rays, where two rays are considered equivalent if their intersection is also a ray. The set of all ends of an infinite graph Γ is called the boundary of Γ and is denoted by $\partial\Gamma$. The distance between two vertices $u, v \in V\Gamma$ for some graph Γ will be denoted by $d_\Gamma(u, v)$ and is defined by the number of edges on a shortest path between the two vertices u and v. We will drop the subscript and merely write $d(u, v)$ if it is clear from the context what graph we are measuring the distance in. For $v \in V\Gamma$, we define the *ball* and *sphere* of radius n as $B(v,n) = \{u \in V\Gamma | d_\Gamma(u,v) \leqslant n\}$ and $S(v,n) = \{u \in V\Gamma | d_\Gamma(u,v) = n\}$ respectively.

Throughout this article we will mainly be discussing infinite locally-finite trees and groups acting on them in a certain ways. We will denote the group of all graph automorphisms of the graph Γ by $\mathrm{Aut}(\Gamma)$.

Given a group G acting on a graph Γ, for any subset $Y \subseteq \Gamma$, G_Y denotes the *stabiliser* subgroup of Y under the action of G, that is, the subgroup of G consisting of all elements $g \in G$ satisfying $gY = Y$. If $Y = \{y\}$ is a singleton set, we will just write G_y instead of $G_{\{y\}}$. Similarly, $\mathrm{Fix}_G(Y)$ denotes the *fixator* of the set Y under the action of G, the subgroup of all elements $g \in G$ satisfying $gy = y$ for all $y \in Y$. The notation $\mathrm{Sym}(X)$ will be used throughout to denote group of all permutation of the set X.

2.2. Totally Disconnected Locally Compact Groups

A *topological group* is a group G with a topology such that the maps:

$$G \times G \to G, (a,b) \mapsto ab$$

$$G \to G, a \mapsto a^{-1}$$

are continuous with respect to the topology. It is easy to check from the definition of a topological group, that the map $G \to G, h \mapsto gh$ for a fixed $g \in G$ is a homeomorphism of G. As a result of this, topological properties of topological groups are typically determined by what happens in a neighbourhood of the identity. A totally disconnected locally compact groups is a topological group whose topology is totally disconnected, that is, the connected components are singleton sets, or equivalently, the connected component of the identity is a singleton, and locally compact, meaning there is a compact neighbourhood of the identity. We will abbreviate totally disconnected locally compact as t.d.l.c. throughout this article.

As already discussed in the introduction, very little was known about t.d.l.c. groups for quite some time. Until the 90's, the only major result known about t.d.l.c. groups was the following theorem by van Dantzig from 1936 [**vD36**] which is referred to as *van Dantzig's Theorem*:

THEOREM 2.1 (van Dantzig). *A totally disconnected group admits a basis at the identity of compact open subgroups*

Since the basis of a topological group is determined by a basis at the identity, this means that every t.d.l.c. group admits a basis of compact open subgroups which are cosets of the above compact open subgroups. As a remark, every compact open subgroup of a t.d.l.c. group is a compact totally disconnected group, also known as a *profinite group*, that is, an inverse limit of finite groups. Profinite groups, which posses many similar properties to finite groups, are well understood examples of t.d.l.c. groups, for example, see [**RZ10, Wil99**] for more details.

As mentioned in the introduction, an important part of the current study of t.d.l.c. groups is the study of automorphism groups of connected infinite locally finite graphs. Given a graph Γ, we can endow its automorphism group $\mathrm{Aut}(\Gamma)$ with the permutation topology. The *permutation topology* is defined as having basis of open sets $\mathcal{B} = \{\mathcal{U}(g, F) \mid g \in \mathrm{Aut}(\Gamma), F \subseteq V\Gamma \text{ finite}\}$ where $\mathcal{U}(g, F) = \{h \in \mathrm{Aut}(\Gamma) \mid g(v) = h(v) \text{ for all } v \in F\}$. We remark that the permutation topology also agrees with the topology of uniform convergence on compact sets and the compact open topology on $\mathrm{Aut}(\Gamma)$ more typically seen in an introductory topology course. When Γ is connected and locally finite, $\mathrm{Aut}(\Gamma)$ becomes a topological group, and infact a t.d.l.c. group as we will soon show:

Proposition 2.2. *Let Γ be a locally finite connected graph. Then $\mathrm{Aut}(\Gamma)$ is a topological group with the permutation topology.*

PROOF. We just need to show that the product and inversion maps are continuous. First we show that the product map is continuous. To do this, let $\alpha, \beta \in \mathrm{Aut}(\Gamma)$ and $F \subseteq V\Gamma$ a finite subset of vertices. It is suffice to show that there exists $F', F'' \subseteq V\Gamma$ finite such that $\mathcal{U}(\alpha, F')\mathcal{U}(\beta, F'') \subseteq \mathcal{U}(\alpha\beta, F)$. Clearly, taking $F' = \beta(F)$ and $F'' = F$ satisfies this. Hence the product map is continuous.

Similarly, to show that the inversion map is continuous, given $F \subseteq V\Gamma$ finite, we need to show that there exists $F' \subseteq V\Gamma$ finite such that $\mathcal{U}(\alpha, F')^{-1} \subseteq \mathcal{U}(\alpha^{-1}, F)$.

Taking $F' = \alpha^{-1}(F)$ does the job, since if $\beta \in \mathcal{U}(\alpha, F')$ i.e. β agrees with α on $\alpha^{-1}(F)$, then β^{-1} must agree with α^{-1} on F i.e. $\beta^{-1} \in \mathcal{U}(\alpha^{-1}, F)$. □

With this topology, the vertex stabilisers in the automorphism group of Γ become compact open subgroups of the automorphism group:

Proposition 2.3. *Let Γ be a locally finite connected graph. For any $v \in V\Gamma$, $\mathrm{Aut}(\Gamma)_v$ is a compact open subgroup of $\mathrm{Aut}(\Gamma)$*

Proof. We will just provide a sketch of the proof. Clearly $\mathrm{Aut}(\Gamma)_v$ is open since it is precisely the open neighbourhood $\mathcal{U}(\mathrm{id}, \{v\})$. It just needs to be shown that $\mathrm{Aut}(\Gamma)_v$ is compact. Fix $v \in V\Gamma$ and take the group of permutations $\mathrm{Sym}(S(v,n))$. This is a compact group for each n since $S(v,n)$ is finite. Define a map $\Phi : \mathrm{Aut}(\Gamma)_v \to \prod_{n\geqslant 1} \mathrm{Sym}(S(v,n)), \alpha \mapsto \prod_{n\geqslant 1} \alpha|_{S(v,n)}$. Clearly this map is injective, since if $\alpha|_{S(v,n)} = \mathrm{id}$ for each n, then we must have $\alpha = \mathrm{id}$.

Now, the image of Φ is closed in $\prod_{n\geqslant 1} \mathrm{Sym}(S(v,n))$, hence is a compact subgroup of $\prod_{n\geqslant 1} \mathrm{Sym}(S(v,n))$ since $\prod_{n\geqslant 1} \mathrm{Sym}(S(v,n))$ is compact by Tychonoff's theorem. Let Φ' be the map Φ with its codomain restricted to the image of Φ. This is a bijection and it can be checked that it is an open map using the fact that $\mathrm{Sym}(S(v,n))$ has the discrete topology. Since every continuous bijection from a compact space to a Hausdorff space is a homeomorphism, we see that the inverse of Φ' is a homeomorphism and hence so is Φ'. This completes the proof. □

As a result of the above proposition, we can now prove the following:

Proposition 2.4. *Let Γ be a locally finite connected graph. Then $\mathrm{Aut}(\Gamma)$ is a totally disconnected locally compact group.*

Proof. It is easy to see that $\mathrm{Aut}(\Gamma)$ is locally compact using the previous proposition: if $\alpha \in \mathrm{Aut}(\Gamma)$ and $v \in V\Gamma$, then $\alpha\mathrm{Aut}(\Gamma)_v$ is a compact neighbourhood of α.

We just need to show that $\mathrm{Aut}(\Gamma)$ is totally disconnected, and to do so, we will show that the connected component of the identity is a singleton. It suffices to show that for any open neighbourhood U of the identity and $\mathrm{id} \neq g \in U$, we can write U as the union of two disjoint open sets U_1 and U_2 such that $\mathrm{id} \in U_1$ and $g \in U_2$.

Let F be a finite subset of $V\Gamma$ such that $\mathcal{U}(\mathrm{id}, F) \subseteq U$. We may assume that $g \notin \mathcal{U}(\mathrm{id}, F)$, since if this is not the case, we can replace F with $F \cup \{v\}$ where $v \in V\Gamma$ and $g(v) \neq v$ and the property will then be satisfied. Set $U_1 = \mathcal{U}(\mathrm{id}, F)$. We will construct U_2 so that $U_1 \cap U_2 = \varnothing$ and $U = U_1 \cup U_2$. For each $h \in U \backslash U_1$, $\mathcal{U}(h, F) \cap U$ is an open neighbourhood of h contained in U disjoint from U_1. Then $U_2 = \bigcup_{h \in U \backslash U_1} (\mathcal{U}(h, F) \cap U)$ satisfies the required properties for U_2. This completes the proof showing $\mathrm{Aut}(\Gamma)$ is totally disconnected.

□

Another result that will be useful to us later in the next chapter is the following, which gives a criteria to determine when the permutation topology on $\mathrm{Aut}(\Gamma)$ or one of its subrgoups is non-discrete:

Proposition 2.5. *Let Γ be a locally finite connected graph and $G \leqslant \mathrm{Aut}(\Gamma)$ a closed subgroup with the subspace topology. The following are equivalent:*

(i) The topology on G is non-discrete.
(ii) G_v is infinite for any $v \in V\Gamma$.
(iii) For every $v \in V\Gamma$ and $n \in \mathbb{N}$, there exists an automorphism $g \in G$ such that $g \mid_{B(v,n)} = \mathrm{id}$ and $g \mid_{B(v,n+1)} \neq \mathrm{id}$.

Proof. Clearly $(iii) \implies (ii)$. Now suppose that G is discrete. Then, for $v \in V\Gamma$, by Proposition 2.3, G_v is a compact open subgroup of G, and it is also discrete since G is. Since every discrete compact space must be finite, we see that G_v is finite. This shows that $(ii) \implies (i)$ by proving the contrapositive. To show that $(i) \implies (iii)$ we also prove the contrapositive. So suppose that (iii) does not hold. Then there exists a vertex $v \in V\Gamma$ and an $n \in \mathbb{N}$ such that any element of G that fixes $B(v, n)$ is the identity. Then $\mathcal{U}(\mathrm{id}, B(v, n))$ is an open set in G that only contains the identity. Since tranlations are continuous in a topological group, we see that all the singleton sets are open in G and hence G is discrete. □

CHAPTER 3

Groups Acting on Trees

In this chapter we aim to give the reader an overview of a number of different examples of (non-discrete) compactly generated totally disconnected locally compact groups acting on trees that are found in the literature. The section starts by introducing the notion of a label-regular tree and their automorphisms groups which were motivated by concepts studied in Tits' well known paper [**Tit70**]. We then proceed to discuss the class of universal groups $U(F)$ and some of their properties, along with the Le Boudec groups $G(F, F')$, which are a generalisation of the universal groups. The idea of a k-closure of an automorphism group of a tree introduced by Banks-Elder-Willis in [**BEW15**] is discussed and we provide a proof of a generalisation of Tits' simplicity theorem here. We also briefly mention some k-closure analogues for universal groups. The chapter concludes with an overview of Neretin's groups and some of their properties.

3.1. Label-regular Trees

Let $\mathcal{T}$ be an infinite locally finite tree without leaves and Ω be a collection of *labels* of possibly infinite cardinality. Fix a labelling $\lambda : V\mathcal{T} \to \Omega$ of the vertices of the tree $\mathcal{T}$. For $v \in V\mathcal{T}$, let $N(v)$ denote the set of all vertices adjacent to v in $\mathcal{T}$ and define a multiset $L(v) := \{\lambda(w) : w \in N(v)\}$. We say that $\mathcal{T}$ is a *label-regular tree* if the multiset $L(v)$ depends only on the value of $\lambda(v)$, that is, if $v_1, v_2 \in V\mathcal{T}$ satisfy $\lambda(v_1) = \lambda(v_2)$, then v_1 and v_2 both have the same number of neighbours with each label. Following the terminology as used in [**Tit70**], the labelling is said to be *normal* if λ is surjective and the group of label preserving automorphisms act transitively on the sets $\lambda^{-1}(\omega)$ for $\omega \in \Omega$.

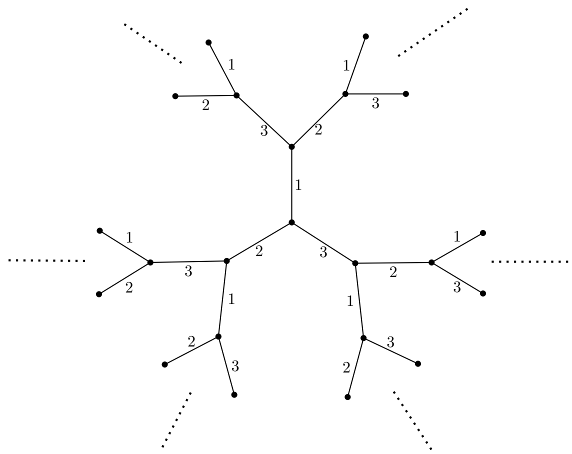
Every label-regular tree is determined by a $|\Omega| \times |\Omega|$ matrix in the following sense: let $\mathcal{T}$ be a label-regular tree with normal labelling and let a_{ij} denote the number of vertices of label j adjacent to a vertex of label i, for $i, j \in \Omega$. Then $\mathbf{a} = (a_{ij})_{i,j\in\Omega}$ is an $|\Omega| \times |\Omega|$ matrix where each of the a_{ij} are non-negative integers and $a_{ij} = 0$ if and only if $a_{ji} = 0$. This matrix determines $\mathcal{T}$ up to isomorphism, furthermore, the graph $G_{\mathbf{a}}$ that has Ω as its vertex set and $\{i, j\}$ is an edge if $a_{ij} \neq 0$, is connected. Conversely, for any matrix $\mathbf{a} = (a_{ij})_{i,j\in\Omega}$ such that each of the a_{ij} are non-negative integers, $a_{ij} = 0$ if and only if $a_{ji} = 0$, and graph $G_{\mathbf{a}}$ connected, there is a label-regular tree denoted $\mathcal{T}_{\mathbf{a}}$ with labels in Ω and such that every vertex of label i has a_{ij} neighbours of label j. Throughout the article, whenever we use the notation $\mathcal{T}_{\mathbf{a}}$, we will assume that $\mathcal{T}_{\mathbf{a}}$ is a label-regular tree and $\mathbf{a}$ is a square matrix that determines the labelling on $\mathcal{T}_{\mathbf{a}}$.

The group of automorphisms $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$ of $\mathcal{T}_{\mathbf{a}}$ is defined to be the group of all automorphisms of the underlying tree that also preserve the labelling i.e. all the automorphisms φ of the underlying tree that satisfy $\lambda(\varphi(v)) = \lambda(v)$ for all $v \in V\mathcal{T}_{\mathbf{a}}$. We will call a sequence of labels $(\omega_i)_{i\in I} \subseteq \Omega$, for some at most countable indexing set I, *compatible* with the labelling on the tree $\mathcal{T}_{\mathbf{a}}$ if there exists a path $(v_i)_{i\in I} \subseteq V\mathcal{T}_{\mathbf{a}}$ satisfying $\lambda(v_i) = \omega_i$ for each $i \in I$.

We will discuss automorphism groups of label-regular trees in a bit more detail in the next chapter when we start looking at Cartan-like decompositions, but for now we move on to looking at some other types of t.d.l.c. groups acting on trees.

3.2. Groups Acting on Trees with Prescribed Local Action

Take the regular tree $\mathcal{T}_d$ of valency d and a set $\Omega = \{1, 2, \ldots, d\}$ of d labels. For $v \in V\mathcal{T}_d$, let $E(v)$ denote the set of edges in $\mathcal{T}_d$ incident with v. At each vertex $v \in V\mathcal{T}_d$, assign a bijective labelling $\lambda_v : E(v) \to \Omega$ with the following property: if $v, w \in V\mathcal{T}_d$ are adjacent vertices and e is the edge connecting v to w, then $\lambda_v(e) = \lambda_w(e)$. Then the labelling $\lambda : E\mathcal{T}_d \to \Omega$ defined such that $\lambda|_{E(v)} = \lambda_v$ for each $v \in V\mathcal{T}_d$ is a well defined labelling of the regular tree $\mathcal{T}_d$ called the *legal labelling* of $\mathcal{T}_d$. Each edge $e \in E\mathcal{T}_d$ is assigned a unique label $\lambda(e)$, and each vertex in $\mathcal{T}_d$ is incident with an edge of each label in Ω. For example, pictured below is a ball of radius 3 in the 3-regular tree with a legal labelling:



Now, given any automorphism $\alpha \in \mathrm{Aut}(\mathcal{T}_d)$, notice that at each vertex $v \in V\mathcal{T}_d$, α induces a permutation of the labels of the edges incident with v: for any edge $e \in E(v)$, the label $\lambda_v(e)$ is sent to $\lambda_{\alpha(v)}(\alpha(e))$. This is clearly a permutation of the labels of the edges in $E(v)$ by definition of the labelling and using the fact that α is an automorphism. Let $\sigma(\alpha, v) \in \mathrm{Sym}(d)$ denote this permutation induced by α on the labels of the edges in $E(v)$. The permutation $\sigma(\alpha, v)$ can be defined explicitly as:

$$\sigma(\alpha, v) := \lambda_{\alpha(v)} \circ \alpha \circ \lambda_v^{-1}$$

This permutation is referred to as the *local action* of α at the vertex v. We may now define the notion of a universal group:

Definition 3.1 (Universal Group)**.** Let $\lambda : E\mathcal{T}_d \to \Omega$ be a legal labelling of $\mathcal{T}_d$ and $F \leqslant \mathrm{Sym}(d)$. The *universal group* on F with respect to the labelling λ, denoted $U^{(\lambda)}(F)$, is defined as $U^{(\lambda)}(F) = \{\alpha \in \mathrm{Aut}(\mathcal{T}_d) \mid \sigma(\alpha, v) \in F \text{ for all } v \in V\mathcal{T}_d\}$.

We remark that if we have two distinct legal labellings $\lambda_1 : E\mathcal{T}_d \to \Omega$ and $\lambda_2 : E\mathcal{T}_d \to \Omega$ of $\mathcal{T}_d$, it can be shown that the universal groups $U^{(\lambda_1)}(F)$ and $U^{(\lambda_2)}(F)$ are conjugate as subgroups of $\mathrm{Aut}(\mathcal{T}_d)$ and hence isomorphic. Thus the group $U^{(\lambda)}(F)$ does not depend on the choice of legal labelling λ, so from now on we will just refer to the universal groups as $U(F)$ and not depending on a specific legal labelling.

We say that the local action of the automorphisms in $U(F)$ is *prescribed* by F. It may not be immediately obvious that a universal group is a group: this follows from the fact that $\sigma(\alpha\beta, v) = \sigma(\alpha, \beta(v))\sigma(\beta, v)$ and $\sigma(\alpha^{-1}, v) = \sigma(\alpha, \alpha^{-1}(v))^{-1}$ which the reader may like to check.

Now, a group of automorphisms $H \leqslant \mathrm{Aut}(\mathcal{T}_d)$ is called *locally permutationally isomorphic* to $F \leqslant \mathrm{Sym}(d)$, if for every $v \in V\mathcal{T}_d$, the action of H_v on $E(v)$ is isomorphic to the action of F on $\{1, 2, \ldots, d\}$. The term 'universal' then comes from the fact that the universal group $U(F)$ is the largest closed subgroup of $\mathrm{Aut}(\mathcal{T}_d)$ that is locally permutationally isomorphic to F (up to isomorphism). For a proof of this fact, see [**CM18**, Proposition 6.23].

The following proposition summarises some of the basic properties of universal groups:

Proposition 3.2. *Let $F \leqslant \mathrm{Sym}(d)$. The following properties hold:*

(i) $U(F)$ is closed in $\mathrm{Aut}(\mathcal{T}_d)$.
(ii) $U(F)$ acts transitively on the vertices of $\mathcal{T}_d$
(iii) $U(F)$ acts transitively on the edges if and only if F acts transitively on Ω.
(iv) $U(F)$ is discrete if and only if F acts freely on Ω.
(v) $U(F)$ is compactly generated.

Proof. To prove (i), we show that the complement $\mathrm{Aut}(\mathcal{T}_d)\backslash U(F)$ is open. Let $g \in \mathrm{Aut}(\mathcal{T}_d)\backslash U(F)$. Then there exists a vertex $v \in V\mathcal{T}_d$ such that $\sigma(g, v) \notin F$. Note that any automorphism in $\mathrm{Aut}(\mathcal{T}_d)$ that agrees with g on $N(v)$ is not contained in $U(F)$ since it has the same local action as g at v. Thus $\mathcal{U}(g, N(v))$ is an open set containing g and is contained in $\mathrm{Aut}(\mathcal{T}_d)\backslash U(F)$. Since this holds for every $g \in \mathrm{Aut}(\mathcal{T}_d)\backslash U(F)$, we see that $\mathrm{Aut}(\mathcal{T}_d)\backslash U(F)$ is open. Thus $U(F)$ must be closed.

(ii): Let $v, v' \in \mathcal{T}_d$. We construct inductively an automorphism $\alpha \in U(F)$ such that $\alpha(v) = v'$. First define α as mapping v to v'. Then for every $u \in N(v)$, define $\alpha(u)$ as the unique vertex adjacent to $\alpha(v)$ such that $\lambda(\{v, u\}) = \lambda(\{\alpha(v), \alpha(u)\})$. Now suppose that α has been defined on $B(v, n)$ for some $n \in \mathbb{N}$. Given $w \in V\mathcal{T}_d$ at distance $n + 1$ from v, let w' be the unique vertex adjacent to w on the path between v and w. Define $\alpha(w)$ to be the unique vertex adjacent to $\alpha(w')$ such that $\lambda(\{w', w\}) = \lambda(\{\alpha(w'), \alpha(w)\})$. Such a vertex exists since there is a bijection between the labels on the edges adjacent to $\alpha(w')$ and the labels on the edges adjacent to w'. Thus we have defined an automorphism α mapping v to v' whose local action at each vertex is the identity. This proves (ii).

(iii): First suppose that $U(F)$ acts transitively on the edges of $\mathcal{T}_d$. Let $\omega_1, \omega_2 \in \Omega$ and fix a vertex $v \in V\mathcal{T}_d$. Let $e_1, e_2 \in E(v)$ such that $\lambda(e_i) = \omega_i$ for $i = 1, 2$. By assumption, there exists an automorphism $\alpha \in U(F)$ such that $\alpha(e_1) = \alpha(e_2)$, in particular, $\sigma(\alpha, v)(\lambda(e_1)) = \lambda(e_2)$. Thus $\sigma(\alpha, v)$ is an element of F and maps ω_1 to ω_2. Since ω_1 and ω_2 were arbitrary, we see that F is transitive on Ω.

Conversely, suppose that the action of F is transitive on the set of labels, and let $e, e' \in E\mathcal{T}_d$ be two distinct edges. Suppose $e = \{v, w\}$ and $e' = \{v', w'\}$ for some $v, v', w, w' \in V\mathcal{T}_d$. If $\lambda(e) = \lambda(e')$, by the proof of (ii), there exists an $\alpha \in U(F)$ such that $\alpha(v) = v'$ and $\lambda(e) = \lambda(\alpha(e))$ for all $e \in E\mathcal{T}_d$, in particular, we must have that $\alpha(e) = e'$. So assume that $\lambda(e) \neq \lambda(e')$. We show that there is a $\beta \in U(F)_{v'}$ such that $\beta(\alpha(e)) = e'$, where $\alpha \in U(F)$ is the automorphism constructed in the proof of (ii) mapping v to v' and preserving the labelling. We define β inductively. First set $\beta(v') = v'$. Let $\sigma \in F$ be such that $\sigma(\lambda(e)) = \lambda(e')$. Define β on $N(v)$ so that $\sigma(\beta, v) = \sigma$. Then β may be extended inductively to the whole of $\mathcal{T}_d$ so that $\sigma(\beta, v) = \sigma$ for all $v \in V\mathcal{T}_d$. It then follows that $\beta \circ \alpha \in U(F)$ and $(\beta \circ \alpha)(e) = e'$.

(iv): First suppose that the action of F is not free. Then there exists an $\omega \in \Omega$ and a non-trivial element $g \in F_\omega$. Now fix a vertex $v \in V\mathcal{T}_d$ and choose an infinite path $\mathcal{P} = (v_i)_{i=1}^{\infty}$ in $\mathcal{T}_d$ starting at v that contains an infinite number of edges of label m. Define $\alpha_n \in U(F)$ $(n \in \mathbb{N})$ to be the automorphism that fixes v and such that the local action at the first n vertices of $\mathcal{P}$, incident to an edge of label ω, is g, and the local action is trivial for the remainder of the vertices on $\mathcal{P}$. Then $\{\alpha_n\}_{n=1}^{\infty}$ is an infinite sequence of distinct automorphisms in $U(F)_v$ and so $U(F)$ is non-discrete by Proposition 2.5.

Conversely suppose that $U(F)$ is non-discrete. By Proposition 2.5 there exists an $\alpha \in U(F)_v$ such that $\alpha \mid_{B(v,n)} = \text{id}$ and $\alpha \mid_{B(v,n+1)} \neq \text{id}$. Thus there exists a $x \in S(v, n)$ such that $\sigma(\alpha, x) \neq \text{id}$ but $\sigma(\alpha, x)$ has a fixed point, hence, the action of F is not free. This completes the proof of (iv).

(v) We will just give an outline of the proof. First, fix a vertex $v \in V\mathcal{T}_d$ and let $e_1, e_2, \ldots, e_d \in E\mathcal{T}_d$ be the edges incident with v such that e_i is the unique such edge with label i. Then for each e_i, there is a unique automorphism $\alpha_i \in U(F)$ that inverts the edge e_i while preserving the labelling of the tree. Now, each of the α_i are contained in $U(\{\mathrm{id}\})$. It can be checked, with the help of the ping-pong lemma from geometric group theory, that $U(\{\mathrm{id}\}) \cong \langle\alpha_1\rangle * \langle\alpha_2\rangle * \cdots * \langle\alpha_d\rangle$. Then, we claim that the set $U(F)_v \cup \{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ generates $U(F)$. Indeed, let $\alpha \in U(F)$ and choose $\beta \in U(\{\mathrm{id}\})$ such that $\beta\alpha(v) = v$; such an element exists since $U(\{\mathrm{id}\})$ is vertex-transitive. Then $\beta\alpha \in U(F)_v$ and it follows that α is in the group generated by $U(F)_v \cup \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$. Since $U(F)_v \cup \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ is the union of two compact sets, it is compact, hence we see that $U(F)$ is compactly generated. □

As mentioned in the introduction to this article, we are particularly interested in studying and understanding the structure of totally disconnected locally compact groups, and understanding particular examples of totally disconnected locally compact groups such as automorphism groups of trees is an important part of the theory. The universal groups $U(F)$ also form another example of a class of totally disconnected locally compact groups since they are a closed subgroup of the totally disconnected locally compact group $\mathrm{Aut}(\mathcal{T}_d)$. We summarise the results so far in the following proposition:

Proposition 3.3. *Let $F \leqslant \mathrm{Sym}(d)$. The group $U(F) \leqslant \mathrm{Aut}(\mathcal{T}_d)$ is a compactly generated, totally disconnected, locally compact Hausdorff topological group. Furthermore, $U(F)$ is discrete if and only if F acts freely on $\mathcal{T}_d$.*

3.2.1. Simplicity Results for Universal Groups.

Here we give an outline of some simplicity results for universal groups. We first start by recalling some work of Tits in [**Tit70**] that gives a condition for when a group of automorphisms acting on a tree is simple. First we describe Tits' independence property which is a vital part of understanding Tits' simplicity theorem. Tits' independence property is often referred to as *Property P* in the literature.

Start by letting G be a group of automorphisms of an infinite locally finite tree $\mathcal{T}$ without leaves, and $\mathcal{P}$ be a path in the tree $\mathcal{T}$ of either finite or infinite length.

Define a function $\pi_{\mathcal{P}} : V\mathcal{T} \to V\mathcal{P}$ such that $\pi_{\mathcal{P}}(v)$, for $v \in V\mathcal{T}$, is the unique closest vertex on the path $\mathcal{P}$ to v. For each vertex v on the path $\mathcal{P}$, let F_v be the restriction of $\mathrm{Fix}_G(\mathcal{P})$ to the subtree $\pi_{\mathcal{P}}^{-1}(v)$. Then there is a natural map:

$$\Phi_{\mathcal{P}} : \mathrm{Fix}_G(\mathcal{P}) \hookrightarrow \prod_{v \in V\mathcal{P}} F_v$$

which essentially describes an automorphism in $\mathrm{Fix}_G(\mathcal{P})$ by what it does on the subtrees $\pi_{\mathcal{P}}^{-1}(v)$ $(v \in V\mathcal{T})$. We say that the group G has Property P, or Tits' Independence Property, if the map $\Phi_{\mathcal{P}}$ given above is an isomorphism for every finite or infinite path $\mathcal{P}$ in $\mathcal{T}$.

Denote by G^+ the subgroup of G generated by the fixators of edges in $\mathcal{T}$ i.e. $G^+ = \langle G_e \mid e \in V\mathcal{T} \rangle$. The following Theorem was proven by Tits' in his article [**Tit70**]; we do not give a proof of the result here, though, a more general result is proven later in this chapter. This Theorem is often referred to as Tits' Simplicity Theorem.

THEOREM 3.4. *Let $\mathcal{T}$ be a tree and G a subgroup of $\mathrm{Aut}(\mathcal{T})$. Suppose that G does not stabilise any non-empty subtree or fix an end of $\mathcal{T}$, and satisfies Property P. Then every non-trivial subgroup of G normalised by G^+ contains G^+, and in particular, G^+ is either simple or trivial.*

It is easy to show that the universal groups $U(F)$ do not stabilise any non-empty subtree or fix any end of $\mathcal{T}_d$ and satisfy Property P. As a result of Tits' Simplicity Theorem, the following result can be deduced about the simplicity of subgroups of universal groups, which was first stated in the original paper on universal groups by Burger and Mozes [**BM00**].

THEOREM 3.5. *Let $F \leqslant \mathrm{Sym}(d)$. The following results hold:*

(i) The group $U(F)^+$ is either simple or trivial

(ii) $U(F)^+$ has finite index in $U(F)$ if and only if F is transitive and generated by point stabilisers, and in this case, $U(F)^+ = U(F) \cap \mathrm{Aut}(\mathcal{T}_d)^+$ and has index 2

3.3. Groups Acting on Trees with Almost Prescribed Local Action

In this section we give a recount of the groups studied by Le Boudec in his paper [**LB16**]. These groups, often referred to as Le Boudec's groups, are groups of automorphisms of a regular tree which generalise the universal groups construction: the requirement that the local actions of each of the automorphisms have to be contained in the subgroup F is slightly weakened. The definition of Le Boudec's groups is given as follows:

Definition 3.6. Let $F, F' \leqslant \mathrm{Sym}(d)$ such that $F \leqslant F'$. Retaining the notation used in the previous section, define the groups $G(F) = \{\alpha \in \mathrm{Aut}(\mathcal{T}_d) \mid \sigma(\alpha, v) \in F \text{ for all but finitely many } v \in V\mathcal{T}_d\}$ and $G(F, F') = G(F) \cap U(F')$.

The group $G(F, F')$ is precisely the group of all automorphisms of $\mathcal{T}_d$ such that $\sigma(\alpha, v)$ is in F' for all $v \in V\mathcal{T}_d$ and in F for all but finitely many $v \in V\mathcal{T}_d$. These groups can be thought of as a 'relaxation' of the requirements for the local action in comparison to the universal groups. In the following, for $\alpha \in G(F, F')$, we will say that a vertex $v \in V\mathcal{T}_d$ is a *singularity* of α if $\sigma(\alpha, v) \notin F$. The collection of all singularities of α is denoted by $S(\alpha)$.

The attentive reader may have already noticed that the groups $G(F, F')$ in many cases will not be closed, and hence also not open in $\mathrm{Aut}(\mathcal{T}_d)$. Thus we do not give these groups the subspace topology from $\mathrm{Aut}(\mathcal{T}_d)$. To understand the topology on these groups, we prove the following result:

Proposition 3.7. *Let G be an abstract group with a topological group H as a subgroup. Then G admits a unique group topology such that the inclusion map $H \hookrightarrow G$ is continuous and open provided that for all open sets $U \subseteq H$, $gUg' \cap H$ is open in H for all $g, g' \in G$.*

Proof. We show that the topology $\mathcal{T}$ on G generated by the left G-translates of open sets in H satisfies the desired properties. It is easy to see that the left G-translates of open sets in H form a basis for a topology on G and the inclusion map is continuous and open with respect to this topology. It just needs to be shown that the multiplication and inversion maps in G are continuous with respect to this topology.

To do this, we first show that the translations maps $L_g : G \to G, x \mapsto gx$ and $R_g : G \to G, x \mapsto xg^{-1}$ for $g \in G$ are homeomorphisms. It is clear that these maps are bijections since $L_{g^{-1}}$ is the inverse of L_g and similarly for R_g. Thus it suffices to show that the maps L_g and R_g are open for any $g \in G$. It is clear that L_g is an open map by definition of the topology. To show that R_g is also open, note that for any open set $U \subseteq H$, for $g' \in G$, Ug' can be written as:

$$Ug' = \bigcup_{g \in G} gH \cap Ug' = \bigcup_{g \in G} g(H \cap g^{-1}Ug')$$

which is a union of left translates of open sets of H and hence is open in the given topology on G. Thus it follows that R_g is also open and hence L_g and R_g are homeomorphisms. To complete the proof of the proposition, it will now be shown that the multiplication and inversion maps in G are continuous.

Let $(g_i)_{i \in I}$ and $(g'_i)_{i \in I}$ be two nets in G converging to $g, g' \in G$ respectively. Then since gH and Hg' are open neighbourhoods of g and g' respectively, we may find nets $(h_i)_{i \in I}$ and $(h'_i)_{i \in I}$ both converging to the identity such that $g_i = gh_i$ and $g'_i = h'_i g$ for each i. It then follows that the net $g_i g'_i = gh_i h'_i g'$ converges to gg' since $h_i h'_i$ converges to the identity in H. In a similar fashion, $g_i^{-1} = h_i^{-1} g^{-1}$ converges to g^{-1} in G since h_i^{-1} converges to the identity in H. Thus this shows that the multiplication and inversion maps are continuous and hence G is a topological group with this topology. □

Since the subgroups of the form $U(F)_T$ for some finite subtree $T \subseteq \mathcal{T}_d$ form a neighbourhood basis of the identity in $U(F)$, and for any $g \in \mathrm{Aut}(\mathcal{T}_d)$, $gU(F)_T g^{-1} = U(F)_{g(T)}$, it is easily seen that the groups $G(F, F')$ satisfy the hypotheses of the above proposition with $H = U(F)$.

The groups $G(F, F')$ are then given the topology such that the inclusion map of $U(F)$ into $G(F, F')$ is continuous and open. Thus, the collection of open neighbourhoods of the identity in $U(F)$ form a collection of open neighbourhoods of the identity in $G(F, F')$. In particular, it follows from the results for universal groups discussed in the previous section that $G(F, F')$ is also a totally disconnected locally compact group and is discrete if and only if F acts freely on Ω.

The following lemma shows that even when an automorphism $\alpha \in G(F)$ is not contained in the universal group $U(F)$, the local action of α at each vertex still behaves 'reasonably' nicely. The proof follows the one given in [**LB16**], as will a number of the other results in this section.

Proposition 3.8. *Given any $\alpha \in G(F)$ and $v \in V\mathcal{T}_d$, the permutation $\sigma(\alpha, v)$ stabilises the orbits of F acting on the set $\{1, 2, \ldots, d\}$.*

Proof. Let $\alpha \in G(F)$ and let V_α denote the set of all vertices for which the statement does not hold. Note that every vertex in V_α is a singularity of α. Suppose for a contradiction that V_α is non-empty and let $v \in V_\alpha$. Then there must exist at least two vertices $v_1, v_2 \in V\mathcal{T}_d$ adjacent to v such that $\sigma(\alpha, v)$ sends the labels $\lambda(\{v, v_1\})$ and $\lambda(\{v, v_2\})$ to elements of different orbits. In a similar fashion, there must exist another vertex v_1' adjacent to v_1 and distinct from v such that the label of the edge $\{v_1, v_1'\}$ is sent to a label in a different orbit. Continuing this argument indefinitely shows that V_α must be infinite which contradicts the fact that $\alpha \in G(F)$ since α can have at most finitely many singularities. □

Given a partition of the set Ω, the *Young subgroup* with respect to the partition is defined to be the maximal subgroup of $\mathrm{Sym}(d)$ stabilising the sets in the partition. Given a subgroup $F \leqslant \mathrm{Sym}(d)$, one may consider the Young subgroup associated to the partition of Ω into F-orbits, which we will denote by $\hat{F} \leqslant \mathrm{Sym}(d)$. It is clear that in this case $F \leqslant \hat{F}$. For the remainder of this article we will always assume that the permutation groups F and F' used in the definition of $G(F, F')$ must be contained in $\hat{F}$. There is no loss of generality in doing this, since by the previous lemma, if the local action of an automorphism in $G(F)$ at a particular vertex is not in F, then it must be contained in $\hat{F}$.

We will need the following lemma in the proof of Proposition 3.10:

Lemma 3.9. *Let $F, F' \leqslant \mathrm{Sym}(d)$ such that $F \leqslant F' \leqslant \hat{F}$. Fix $v \in V\mathcal{T}_d$ and $n \in \mathbb{N}$. Suppose that $\beta \in \mathrm{Aut}(\mathcal{T}_d)$ such that $\sigma(\beta, w) \in F'$ for all $w \in B(v, n)$. Then there exists an automorphism $\alpha \in G(F, F')$ such that $\sigma(\alpha, w) \in F$ for all $w \in \mathcal{T}_d \backslash B(v, n)$, and α and β agree on $B(v, n+1)$.*

PROOF. Given $x \in S(v,n)$, let V_x denote the set of vertices $w \in V\mathcal{T}_d$ such that the unique path from v to w passes through x. We may assume that β fixes the vertex v; if not, consider the automorphism $\beta' = \gamma\beta$ where $\gamma \in U(F)$ is the automorphism sending $\beta(v)$ to v and whose local action at every vertex is the identity.

First we define α to fix v. Then α is determined by its local action at each vertex in $\mathcal{T}_d$, thus, it suffices to define $\sigma(\alpha, w)$ for each vertex $w \in V\mathcal{T}_d$. Start by defining $\sigma(\alpha, w) = \sigma(\beta, w)$ for each $w \in B(v,n)$. Then for each vertex $x \in S(v,n)$ and $\omega \in \Omega$, since $F' \leqslant \hat{F}$, choose a permutation $\theta(\omega, x) \in F$ (not necessarily unique) such that $\sigma(\beta, x)(\omega) = \theta(\omega, x)(\omega)$. Then given $w \in V_x$, $x \in S(v,n)$, define $\sigma(\alpha, w)$ to be $\theta(\omega, x)$ where $\omega \in \Omega$ is the unique label on the edge with origin x which lies on the path between x and w. It can be easily seen that this is a well defined automorphism α satisfying the required properties. □

As a result of the preceding lemma, we can now prove the following result:

Proposition 3.10. *Let $F, F' \leqslant \mathrm{Sym}(d)$ such that $F \leqslant F' \leqslant \hat{F}$. Then the following properties hold:*

(i) The closure $\overline{G(F,F')}$ of $G(F,F')$ in $\mathrm{Aut}(\mathcal{T}_d)$ is equal to $U(F')$
(ii) $G(F)$ is dense in $\mathrm{Aut}(\mathcal{T}_d)$ if and only if the action of F on Ω is transitive

PROOF. For (i), first note that $G(F,F') \subseteq U(F')$, and since $U(F')$ is closed in $\mathrm{Aut}(\mathcal{T}_d)$, we must have that $\overline{G(F,F')} \subseteq U(F')$. We just need to show that $U(F') \subseteq \overline{G(F,F')}$. Let $\alpha \in U(F')$. By the previous lemma, for each $n \in \mathbb{N}$, there exists an automorphism $\alpha_n \in G(F,F')$ such that α_n agrees with α on $B(v,n)$ and $\sigma(\alpha_n, w) \in F$ for each $w \in V\mathcal{T}_d \backslash B(v,n)$. Then the sequence $(\alpha_n)_{n=1}^{\infty} \subseteq G(F,F')$ converges to α. Hence $\alpha \in \overline{G(F,F')}$. This completes the proof of (i)

For (ii), the reverse direction follows by applying (i) and using the fact that $\hat{F} = \mathrm{Sym}(d)$ if F is transitive and $U(\mathrm{Sym}(d)) = \mathrm{Aut}(\mathcal{T}_d)$. Conversely, if the action of F is not transitive then $\overline{G(F)} = U(\hat{F})$ is a strict subset of $\mathrm{Aut}(\mathcal{T}_d)$ since $\hat{F}$ is not transitive. □

This proposition confirms the fact mentioned earlier in the section that the groups $G(F, F')$ are not generally closed in $\mathrm{Aut}(\mathcal{T}_d)$. Indeed, $G(F, F')$ is closed if and only if it is equal to the universal group $U(F')$. This will be an interesting fact in relation to material we look at later in the article. Similarly to universal groups, the groups $G(F, F')$ are also compactly generated. We give the proof of this result below, but first we need to prove a short lemma. In the following, for $v \in V\mathcal{T}_d$ and $n \in \mathbb{N}$, $G(F, F')_{(v,n)}$ denotes the set of all automorphisms in $G(F, F')$ who fix the vertex v and have all their singularities contained in $B(v, n)$.

Lemma 3.11. *Let $F, F' \leqslant \mathrm{Sym}(d)$ such that $F \leqslant F' \leqslant \hat{F}$. For any $v \in V\mathcal{T}_d$ and $n \in \mathbb{N}$, the groups $G(F, F')_{(v,n)}$ are compact.*

Proof. For a fixed $v \in V\mathcal{T}_d$, since $U(F)_v$ is a compact open subgroup of $\mathrm{Aut}(\mathcal{T}_d)$, it is also compact open in $G(F, F')_{(v,n)}$ for any $n \in \mathbb{N}$ by definition of the topology on $G(F, F')$. It can also be checked that $U(F)_v$ has finite index in $G(F, F')_{(v,n)}$ for any $n \in \mathbb{N}$. Hence, we can write $G(F, F')_{(v,n)}$ as a finite union of translates of $U(F)_v$. Thus $G(F, F')_{(v,n)}$ is compact open in $G(F, F')$ being the finite union compact open sets. $\square$

Proposition 3.12. *Let $F, F' \leqslant \mathrm{Sym}(d)$ such that $F \leqslant F' \leqslant \hat{F}$. The groups $G(F, F')$ are compactly generated.*

Proof. Given $g \in G(F, F')$ with at most m singularities, we claim that there exists vertices $v_1, v_2, \ldots, v_m \in V\mathcal{T}_d$, $g_i \in G(F, F')_{(v_i,0)}$ for each i, and $h \in U(F)$ such that $g = hg_1g_2 \cdots g_m$. We prove this by induction on m, the number of singularities of g. If $m = 0$, then $g \in U(F)$ and the result is clear. Now suppose that $n \in \mathbb{N}$ and the result holds for all $m < n$, and suppose that g has n singularities. Let $v \in V\mathcal{T}_d$ be a singularity of g. By vertex transitivity of $U(F)$, we may find an element $h \in U(F)$ such that $g' = hg$ fixes the vertex v. Since $\sigma(h, w) \in F$ for all $w \in V\mathcal{T}_d$, it is easy to see that g and g' have the same singularities. By Lemma 3.9, there exists $k \in G(F, F')_{(v,0)}$ such that $\sigma(k, v) = \sigma(g', v)$. Then $g'' = g'k^{-1} = hgk^{-1}$ fixes $B(v, 1)$. Since $k \in G(F, F')_{(v,0)}$, the singularities of g'' are precisely all the singularities of g' different from v. Hence, g'' has at most $n-1$ singularities, and thus by the induction hypothesis, there exists $v_1, v_2, \ldots, v_{n-1} \in V\mathcal{T}_d$, $g_i \in G(F, F')_{(v_i,0)}$

for each i and $h' \in U(F)$ such that $g'' = h' g_1 g_2 \cdots g_{n-1}$. It then follows that $g = h^{-1} h' g_1 g_2 \cdots g_{n-1} k$ is in the desired form.

Now, since $U(F)$ is vertex transitive and $gG(F,F')_{(v,0)}g^{-1} = G(F,F')_{(gv,0)}$ for any $g \in U(F)$, we see that the group generated by $U(F)$ and $G(F,F')_{(v_0,0)}$ for a fixed $v_0 \in V\mathcal{T}_d$ contains $G(F,F')_{(w,0)}$ for every $w \in V\mathcal{T}_d$. By the arguments in the previous paragraph, this means that $U(F) \cup G(F,F')_{(v_0,0)}$ generates $G(F,F')$. Thus if K is a compact generating set for $U(F)$, which exists by Proposition 3.2, then $K \cup G(F,F')_{(v_0,0)}$ is a compact generating set for $G(F,F')$. □

We summarise the results of this section in the following proposition:

Proposition 3.13. *Let $F, F' \leqslant \mathrm{Sym}(d)$ such that $F \leqslant F' \leqslant \hat{F}$. The groups $G(F,F')$ are compactly generated, totally disconnected, locally compact Hausdorff groups. Furthermore, $G(F,F')$ is discrete if and only if F acts freely on Ω.*

Hence, this gives us more examples of (non-discrete) compactly generated, totally disconnected locally compact groups.

3.3.1. A Simplicity Result for the Groups $G(F,F')$. In a similar fashion to the simplicity result for universal groups, there are also some results concerning simplicity of subgroups of Le Boudec's groups $G(F,F')$ under certain assumptions on F and F'. For the case of universal groups, the proof that the subgroups $U(F)^+$ are simple uses the fact that the universal groups satisfy Tits' Property P, and then the result follows directly from Tits' Simplicity Theorem. In a similar way, Le Boudec uses a weaker version of Tits' Property P for the groups $G(F,F')$, which he calls the *edge independence property*. The edge independence property is obtained by restricting the path to be a single edge in the definition of Property P. Using some results concerning the edge-independence property, the following simplicity result can be deduced, which can be found as Theorem 4.13 in [**LB16**]:

Theorem 3.14. *Let $F \leqslant F' \leqslant \mathrm{Sym}(d)$. Suppose that F is transitive and $F' = \langle [F'_\omega, F'_\omega] \cup F_\omega \mid \omega \in \Omega \rangle$. Then the group $G(F,F')^+$ is simple.*

Also, define the group $N(F,F') = \langle [G(F,F')_e, G(F,F')_e] \mid e \in E\mathcal{T}_d \rangle$. Le Boudec also gives a proof of the following result in [**LB16**]:

THEOREM 3.15. *Let $F \leqslant F' \leqslant F'' \leqslant \mathrm{Sym}(d)$ such that F' has index two in F'' and the type-preserving subgroup of $G(F, F'')$ is simple. Then $N(F, F')$ is a simple subgroup of index eight in $G(F, F')$.*

3.4. k-closures of Groups Acting on Trees

In the paper [**BEW15**], Banks-Elder-Willis provide another novel construction of groups acting on trees. Given a group G acting on a tree $\mathcal{T}$, they define, for each $k \in \mathbb{N}$, the k-closure $G^{(k)}$ of G which also acts as a group of automorphisms on $\mathcal{T}$, and in a sense captures the local action of G on balls of radius k in $\mathcal{T}$. Tits' Property P is also generalised in this paper which results in a more general version of Tits' simplicity theorem, and it is shown that under certain assumptions each of the groups $G^{(k)}$ has simple subgroups similar to what we discussed earlier for universal groups. This work on k-closures also motivates a more general definition of universal groups where the local action is prescribed on balls of radius k in $\mathcal{T}$. Here we give an overview of this work.

First we define what is meant by the k-closure of a group; throughout this section we will be assuming that $\mathcal{T}$ is an arbitrary locally finite tree.

Definition 3.16. Let G be a group of automorphisms of a tree $\mathcal{T}$. For fixed $k \in \mathbb{N}$, the k-closure of G, denoted $G^{(k)}$, is defined as:

$$G^{(k)} = \{g \in \mathrm{Aut}(\mathcal{T}) \mid \forall v \in V\mathcal{T}, \exists h_v \in G \text{ such that } g|_{B(v,k)} = h_v|_{B(v,k)}\}$$

The groups $G^{(k)}$ are precisely all the automorphism of the tree $\mathcal{T}$, such that on balls of radius k around each vertex, they agree with an element of G. The k-closures of a group G enjoy the following properties:

Proposition 3.17. *Let $\mathcal{T}$ be a tree and $G \leqslant \mathrm{Aut}(\mathcal{T})$. Then, for any $k \in \mathbb{N}$, the following hold:*

(i) $G \leqslant G^{(k)}$ *for each* $k \in \mathbb{N}$
(ii) $G^{(k)}$ *is a closed subgroup of* $\mathrm{Aut}(\mathcal{T})$
(iii) $G^{(l)} \leqslant G^{(k)}$ *for all* $l \geqslant k$
(iv) $\bigcap_{k\in\mathbb{N}} G^{(k)} = \overline{G}$

(v) The orbit of $v \in V\mathcal{T}$ under the action of $G^{(k)}$ is equal to the orbit of v under the action of G.

Proof. (i): This is clear since every automorphism $g \in G$ agrees with itself on $B(v,k)$ for all $v \in V\mathcal{T}$.

(ii): We will show that the complement of $G^{(k)}$ in $\mathrm{Aut}(\mathcal{T})$ is open. For any $g \in \mathrm{Aut}(\mathcal{T})\backslash G^{(k)}$, there exists a vertex $v \in V\mathcal{T}$ such that g does not agree with any element of G on $B(v,k)$. Thus $\mathcal{U}(g, B(v,k))$ is an open set that does not contain any element of $G^{(k)}$ i.e. $\mathcal{U}(g, B(v,k)) \subseteq \mathrm{Aut}(\mathcal{T})\backslash G^{(k)}$. Since this holds for any $g \in \mathrm{Aut}(\mathcal{T})\backslash G^{(k)}$, we see that $\mathrm{Aut}(\mathcal{T})\backslash G^{(k)}$ is open and hence $G^{(k)}$ is closed.

(iii): Let $g \in G^{(l)}$. For every vertex $v \in V\mathcal{T}$, there exists a $g_v \in G$ such that g agrees with g_v on $B(v,l)$. Then, clearly g agrees with g_v on $B(v,k)$ since $k \leqslant l$. Since this holds for every vertex $v \in V\mathcal{T}$, we see that $g \in G^{(k)}$.

(iv): By (i) $G \subseteq G^{(k)}$ for each k, and by (ii) $G^{(k)}$ is closed, hence $\overline{G} \subseteq G^{(k)}$ for each k. Thus it follows that $\overline{G} \subseteq \bigcap_{k\in\mathbb{N}} G^{(k)}$. So it just remains to show that $\bigcap_{k\in\mathbb{N}} G^{(k)} \subseteq \overline{G}$. To do this, we will show that for any $g \in \bigcap_{k\in\mathbb{N}} G^{(k)}$, every open set of g intersects G non-trivially and hence it follows that $g \in \overline{G}$ i.e. $\bigcap_{k\in\mathbb{N}} G^{(k)} \subseteq \overline{G}$. Indeed, if $g \in \bigcap_{k\in\mathbb{N}} G^{(k)}$ and U is open neighbourhood of g in $\mathrm{Aut}(\mathcal{T})$, then U must contain $\mathcal{U}(g, B(v,k))$ for some $v \in V\mathcal{T}$ and $k \in \mathbb{N}$ since $\mathcal{B} = \{\mathcal{U}(g, B(v,k)) \mid v \in V\mathcal{T}, k \in \mathbb{N}\}$ forms a basis for the topology on $\mathrm{Aut}(\mathcal{T})$. But $\mathcal{U}(g, B(v,k))$ contains an element of G since $g \in G^{(k)}$. Hence U intersects G and this proves (iv).

(v): Since $G \leqslant G^{(k)}$, the orbit Gv is contained in $G^{(k)}v$. Conversely, if $v' \in G^{(k)}v$, then there exists an automorphism $g \in G^{(k)}$ that maps v to v'. Then, by definition of the group $G^{(k)}$, there is an automorphism in G that agrees with g on $B(v,k)$ i.e. there must exist an automorphism in G that maps v to v'. Hence $G^{(k)}v \subseteq Gv$ and the result follows. □

Just like the previous groups we have looked at in this article, the k-closures are non-discrete under certain circumstances. The proof of the following theorem can be found in the paper by Banks-Elder-Willis.

THEOREM 3.18 ([**BEW15**]). *Let $\mathcal{T}$ be a tree and $G \leqslant \mathrm{Aut}\mathcal{T}$. Fix $k \in \mathbb{N}$ and suppose that G does not stabilise any non-empty proper subtree of $\mathcal{T}$. Then $G^{(k)}$ is non-discrete if and only if there is an edge $(v,w) \in E\mathcal{T}$ and $g \in G$ such that:*

$$g \mid_{B(v,k)\cap B(w,k)} = 1 \text{ and } g \mid_{B(w,k)} \neq 1$$

Equivalently, $G^{(k)}$ is discrete if and only if $\mathrm{Fix}_G(B(v,k) \cap B(w,k)) = \{1\}$ for every $(v,w) \in E\mathcal{T}$.

This theorem has a number of corollaries which can be found in [**BEW15**] that give relations between the properties of a group G and its k-closure $G^{(k)}$. We now move on to discuss some independence properties that will be used later in the article.

3.4.1. Independence Properties and Simple Subgroups. Earlier in this chapter we briefly discussed Tits' Property P and saw his simplicity theorem that says that any group acting on a tree satisfying Property P and not stabilising any proper non-empty subtree or fixing an end of the tree has a simple subgroup G^+. Here we give an overview of two other independence properties, Property IP_k and Property P_k, for groups acting on trees, and prove a generalisation of Tits' Simplicity Theorem. These results also have consequences in the context of k-closures of automorphism groups of trees. Once again, these results are from the paper [**BEW15**].

First we give the definition of Property IP_k which is a special case of Property P_k that will be defined shortly:

Definition 3.19 (Property IP_k)**.** Let $\mathcal{T}$ be a tree and $G \leqslant \mathrm{Aut}(\mathcal{T})$. Fix $k \in \mathbb{N}$ and let $e = (v,w) \in E\mathcal{T}$. Define

$$F_{k,e} := \mathrm{Fix}_G(B(v,k) \cap B(w,k)).$$

Then G satisfies Property IP_k if for every edge $e = (v,w) \in E\mathcal{T}$,

$$F_{k,e} = \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$$

where we recall that $\mathcal{T}_{(v,w)}$ and $\mathcal{T}_{(w,v)}$ are the semi-trees containing w and v respectively obtained from $\mathcal{T}$ by removing the edge (v,w).

When $k = 1$, Property IP_k is just Tits' Property P with the path $\mathcal{P}$ replaced by a single edge. This is precisely what Le Boudec called the edge independence property that we discussed in the previous section of this chapter and was used to prove some simplicity results for the groups $G(F, F')$.

We note that the k-closure of a group acting on a tree always satisfies Property IP_k:

Proposition 3.20. *Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ and $k \in \mathbb{N}$. Then $G^{(k)}$ satisfies Property IP_k.*

Proof. Fix an edge $e = (v,w) \in E\mathcal{T}$. Since $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})$ and $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$ both fix $B(v,k) \cap B(w,k)$, it follows that $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$ does to. Hence $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)}) \subseteq F_{k,e}$. Thus we just need to show that $F_{k,e} \subseteq \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$.

Let $g \in F_{k,e}$. Define an automorphism g_1 of $\mathcal{T}$ by $g_1 = g$ on $B(u,k)$ for every $u \in V\mathcal{T}_{(w,v)}$ and trivial on $B(u,k)$ for every $u \in V\mathcal{T}_{(v,w)}$. Then $g_1 \in \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)}) \subseteq G^{(k)}$. Similarly define an automorphism g_2 by $g_2 = g$ on $B(u,k)$ for every $u \in V\mathcal{T}_{(v,w)}$ and trivial on $B(u,k)$ for every $u \in V\mathcal{T}_{(w,v)}$. Then $g_2 \in \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$ and $g = g_1 g_2$. Thus $g \in \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$ which completes the proof. □

We now give the definition of Property P_k, which is a generalisation of Tits' Property P discussed earlier in this Chapter. It will be seen shortly that groups satisfying Property P_k that do not stabilise any proper subtree or fix an end of $\mathcal{T}$ have a simple subgroup G^{+_k} similar to the case of groups satisfying Property P. In the following, given a subtree $T \subseteq \mathcal{T}$, T^k will denote the subtree of $\mathcal{T}$ spanned by the vertices at distance at most k from T.

Definition 3.21 (Property P_k)**.** Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ and $\mathcal{P}$ a path in $\mathcal{T}$ either of finite or (bi-)infinite in length. Let $\pi_{\mathcal{P}}$ be the nearest point projection of $V\mathcal{T}$ onto $V\mathcal{P}$ as defined earlier, and denote by $F_{(k-1,v)}$, for $v \in V\mathcal{P}$, the restriction of $\mathrm{Fix}_G(\mathcal{P}^{k-1})$ to $\pi_{\mathcal{P}}^{-1}(v)$. Then we say that G has *Property P_k* if the canonical homomorphism $\Phi_{\mathcal{P}} : \mathrm{Fix}_G(\mathcal{P}^{k-1}) \hookrightarrow \prod_{v \in V\mathcal{P}} F_{(k-1,v)}$ is an isomorphism for every path $\mathcal{P}$ in $\mathcal{T}$.

We now show that for closed subgroups of $\mathrm{Aut}(\mathcal{T})$, Property P_k is in fact equivalent to Property IP_k. Maintaining the notation used in the previous definition, we first prove the following lemma:

Lemma 3.22. *Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ and suppose that G satisfies Property IP_k. Let $\mathcal{P}$ be a finite path in $\mathcal{T}$. Then the canonical map $\Phi_{\mathcal{P}} : \mathrm{Fix}_G(\mathcal{P}^{k-1}) \hookrightarrow \prod_{v \in V\mathcal{P}} F_{(k-1,v)}$ is an isomorphism.*

PROOF. The proof is by induction on the length of the path $\mathcal{P}$. If the length of $\mathcal{P}$ is one, then since G satisfies Property IP_k, this implies that $\Phi_{\mathcal{P}}$ is an isomorphism by definition.

Now suppose that the result holds for all paths of length less than or equal to $m-1$ and suppose that $\mathcal{P}$ has length m. It is clear that $\Phi_{\mathcal{P}}$ is injective, it just needs to be shown that $\Phi_{\mathcal{P}}$ is surjective. Let $v_1, v_2, \ldots, v_m$ denote the vertices on the path $\mathcal{P}$ and let $\prod_{i=1}^{m} g_i \in \prod_{i=1}^{m} F_{(k-1,v_i)}$ where $g_i \in F_{(k-1,v_i)}$ for $i = 1, \ldots, m$. Let $\mathcal{P}'$ denote the path $\mathcal{P}$ with the vertex v_m and the adjoining edge removed, and $\Phi_{\mathcal{P}'} : \mathrm{Fix}_G((\mathcal{P}')^{k-1}) \to \prod_{v \in V\mathcal{P}'} F_{(k-1,v)}$ the canonical homomorphism.

Define $\tilde{g}$ to agree with g_{m-1} on $\pi_{\mathcal{P}}^{-1}(v_{m-1})$, g_m on $\pi_{\mathcal{P}}^{-1}(v_m)$ and the identity elsewhere. By the induction hypothesis, there exists $g \in \mathrm{Fix}_G((\mathcal{P}')^{k-1})$ such that $\Phi_{\mathcal{P}'}(g) = g_1 g_2 \cdots g_{m-2}\tilde{g}$. Since g agrees with g_m on $\pi^{-1}(v_m)$, g fixes $\mathcal{P}^{k-1}$ and hence $g \in \mathrm{Fix}_G(\mathcal{P}^{k-1})$. Also $\Phi_{\mathcal{P}}(g) = \prod_{i=1}^{m} g_i$ which completes the proof. □

Proposition 3.23. *Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ be a closed subgroup. Then G satisfies Property IP_k if and only if G satisfies Property P_k.*

PROOF. It is clear that if G satisfies Property P_k then G satisfies Property IP_k. So we just need to prove that if G satisfies Property IP_k then it satisfies Property P_k. By the previous lemma, it just remains to be shown that $\Phi_{\mathcal{P}} : \mathrm{Fix}_G(\mathcal{P}^{k-1}) \to \prod_{v \in V\mathcal{P}} F_{(k-1,v)}$ is an isomorphism for every infinite path $\mathcal{P} \subseteq \mathcal{T}$.

So suppose that $\mathcal{P}$ is an infinite path, and lets assume that it is bi-infinite; the proof is essentially the same for when $\mathcal{P}$ is a ray. Let $(v_i)_{i \in \mathbb{Z}}$ be the vertices on the path $\mathcal{P}$ and denote by $\mathcal{P}_n$ the path from v_{-n} to v_n. To prove the proposition, we just need to show that $\Phi_{\mathcal{P}}$ is surjective. Let $\prod_{i \in \mathbb{Z}} g_i \in \prod_{v \in V\mathcal{P}} F_{(k-1,v)}$ where $g_i \in F_{(k-1,v_i)}$ for each i. We need to find $g \in \mathrm{Fix}_G(\mathcal{P}^{k-1})$ such that $\Phi_{\mathcal{P}}(g) = \prod_{i \in \mathbb{Z}} g_i$.

The previous lemma gives is a sequence of elements $\{\tilde{g}_n\}_{n\in\mathbb{N}}$, $\tilde{g}_n \in \mathrm{Fix}_G((\mathcal{P}_n)^{k-1})$ for each n, such that $\Phi_{\mathcal{P}_n}(\tilde{g}_n) = g'_{-n}g_{-n+1}\cdots g_{n-1}g'_n$, where g'_n (resp. g'_{-n}) denotes the automorphism of $\pi^{-1}_{\mathcal{P}_n}(v_n)$ (resp. $\pi^{-1}_{\mathcal{P}_n}(v_{-n})$) that agrees with g_i on $\pi_{\mathcal{P}}(v_i)$ for $i \geqslant n$ (resp. $i \leqslant -n$). By closedness of G, there exists $g \in \mathrm{Fix}_G(\mathcal{P}^{k-1})$ such that $\tilde{g}_n \to g$. Since $\Phi_{\mathcal{P}_n}(\tilde{g}_n)$ agrees with $\Phi_{\mathcal{P}}(g)$ on $\pi^{-1}_{\mathcal{P}}(v_i)$ for $-n \leqslant i \leqslant n$, we must have that $\Phi_{\mathcal{P}_n}(\tilde{g}_n) \to \Phi_{\mathcal{P}}(g)$ as $n \to \infty$. But $\Phi_{\mathcal{P}_n}(\tilde{g}_n) \to \prod_{i\in\mathbb{Z}} g_i$, hence, $\Phi_{\mathcal{P}}(g) = \prod_{i\in\mathbb{Z}} g_i$ which completes the proof. □

Thus, this proposition shows that for closed subgroups of $\mathrm{Aut}(\mathcal{T})$, to check that the subgroup satisfies Property P_k, we just need to check that $\Phi_{\mathcal{P}}$ is an isomorphism whenever the path $\mathcal{P}$ is an edge in $\mathcal{T}$.

3.4.2. An Interlude on Some Work of Tits'. For use in the proof of the simplicity theorem, and for use in later chapters of this article, we recall the following results of Tits' concerning group acting on trees without stabilising any proper non-empty subtree or fixing any end. The first result is Lémmè 4.4 in [**Tit70**]:

Lemma 3.24. *Suppose that N and G are non-trivial subgroups of* $\mathrm{Aut}(\mathcal{T})$ *and N is normalised by G. If G does not stabilise any non-empty subtree or fix any end of $\mathcal{T}$, then the same is true for N.*

The following result is Lémmè 4.1 in [**Tit70**]:

Lemma 3.25. *Let $G \leqslant \mathrm{Aut}(\mathcal{T})$. The following are equivalent:*

(i) G does not stabilise any proper non-empty subtree of $\mathcal{T}$.
(ii) The orbit Gv of any vertex $v \in V\mathcal{T}$ has non-empty intersection with any semi-tree in $\mathcal{T}$.

The following proposition, which can be found as Proposition 3.4 in Tits' article, will also come in handy:

Proposition 3.26. *If $G \leqslant \mathrm{Aut}(\mathcal{T})$ contains no translations, then G is contained in either the stabiliser of a vertex, the stabiliser of an edge or the fixator of an end.*

3.4.3. A Generalisation of Tits' Simplicity Theorem. We now finish the section by giving a proof of a generalised version of Tits' simplicity theorem seen in [**BEW15**]. We will follow the proof given by Banks-Elder-Willis. The proof is essentially repeating the proof of Tits' original simplicity theorem seen in [**Tit70**], however, with this more general k-closure notation substituted. First we prove the following lemma which is an analogue of [**Tit70**, Lémmè 4.3]:

Lemma 3.27. *Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ be a closed subgroup and suppose that $g \in G$ is a translation along a bi-infinite path $\mathcal{P}$. Let K be the fixator of $\mathcal{P}^{k-1}$ in G. Then, if G satisfies Property P_k, $K = [g, K] = \{gkg^{-1}k^{-1} : k \in K\}$.*

Proof. Since g is a translation along $\mathcal{P}$, g stabilises $\mathcal{P}^k$ setwise, hence, $gkg^{-1} \in K$ for all $k \in K$. This means that $[g, K] \subseteq K$. To show the other inclusion, let $k \in K$. We will show that there exists a $k' \in K$ such that $k = gk'g^{-1}k'^{-1}$. Using the notation as in Definition 3.21, and identifying $V\mathcal{P}$ with $\mathbb{Z}$, since G satisfies Property P_k we may write $k = \prod_{i\in\mathbb{Z}} f_i$ where $f_i \in F_{(k-1,i)}$ for each i.

We define k' by finding $k'_i \in F_{(k-1,i)}$ for each i so that $k' = \prod_{i\in\mathbb{Z}} k'_i$ satisfies the required equality. For $i \in \mathbb{Z}$, let $\alpha_i : F_{(k-1,i)} \to F_{(k-1,i+d)}$ be the automorphism induced by conjugating by g, where d denotes the distance that g translates the path $\mathcal{P}$. We define the k'_i inductively. For $0 \leqslant i \leqslant d-1$, choose $k'_i \in F_{(k-1,i)}$ arbitrarily. If $i \geqslant d$ define $k'_i = k_i^{-1}\alpha_{i-d}(k'_{i-d})$ and if $i < 0$ $k'_i = \alpha_i^{-1}(f_{i+d}g_{i+d})$. □

To be used in the following theorem, we make the following definition:

Definition 3.28. Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ and fix $k \in \mathbb{N}$. For $e = \{v, w\} \in E\mathcal{T}_d$ let $F_{k,e} := \mathrm{Fix}_G(B(v,k) \cap B(w,k))$. We define the subgroup G^{+_k} by $G^{+_k} := \langle F_{k,e} \mid e \in E\mathcal{T}\rangle$.

We finally come to the generalisation of Tits' Simplicity Theorem:

Theorem 3.29. *Fix $k \in \mathbb{N}$ and let $G \leqslant \mathrm{Aut}(\mathcal{T})$ such that G does not stabilise any proper non-empty subtree or an end of $\mathcal{T}$, and satisfies Property P_k. Then every nontrivial subgroup of G normalised by G^{+_k} contains G^{+_k}. In particular, G^{+_k} is either simple or trivial.*

Proof. Assume the hypotheses of the Theorem and let H be a nontrivial subgroup of G normalised by G^{+_k}. Since G satisfies Property P_k, for every edge

$e = \{v, w\} \in E\mathcal{T}$, $F_{k,e} := \mathrm{Fix}_G(B(v,k) \cap B(w,k)) = \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}(\mathcal{T}_{(w,v)})$. To prove the theorem, we just need to show that for every edge $e = \{v, w\}$, $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)}) \subseteq H$ and $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)}) \subseteq H$, and it is suffice to solely show that $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)}) \subseteq H$ since the other inclusion will follow from the same argument by interchanging the roles of v and w.

Since G^{+_k} is normal in G, by Lemma 3.24 G^{+_k} does not stabilise any non-empty subtree of $\mathcal{T}$ or fix any end of $\mathcal{T}$. By Lemma 3.24 again, H also satisfies these properties since H is normalised by G^{+_k}. Also, by Proposition 3.26, there is a non-trivial translation $h \in H$. Let $\mathcal{P}$ be the bi-infinite path along which h translates and let $-\infty$ and ∞ denote the ends of $\mathcal{P}$. We claim that $\mathcal{P} \subseteq \mathcal{T}_{(v,w)}$.

By Lemma 3.25, $Hv \cap \mathcal{T}_{(v,w)} \neq \varnothing$ for all $v \in V\mathcal{P}$, hence, there is $g \in H$ with $g(\mathcal{P}) \cap \mathcal{T}_{(v,w)} \neq \varnothing$. Replacing $\mathcal{P}$ and h with $g(\mathcal{P})$ and ghg^{-1} if necessary, we may assume that $\mathcal{P} \cap \mathcal{T}_{(v,w)} \neq \varnothing$. This intersection must be atleast an infinite path, if not, a biinfinite path. Lets suppose that ∞ is the end contained in $\mathcal{P} \cap \mathcal{T}_{(v,w)}$.

Since H does not fix any end of $\mathcal{T}$, we may find $f \in H$ such that $f(-\infty) \notin \{-\infty, \infty\}$, moreover, $f^{-1}(\mathcal{P})$ does not contain any representative of the end $-\infty$. If $\pi : V\mathcal{T} \to V\mathcal{P}$ is the projection of $\mathcal{T}$ onto $\mathcal{P}$, then either $\pi(\mathcal{T}_{(w,v)})$ is a single vertex if $\mathcal{P} \subset \mathcal{T}_{(v,w)}$ or an infinite path that forms a representative for the end $-\infty$. Since $f^{-1}(\mathcal{P})$ does not contain any representative of the end $-\infty$, the projection $\pi(f^{-1}(\mathcal{P}))$ must be contained in some representative of ∞ that is also contained in $\mathcal{P}$.

Let $\tilde{\mathcal{P}}$ be the shortest representative of ∞ such that $\pi(f^{-1}(\mathcal{P})) \subseteq \tilde{\mathcal{P}} \subseteq \mathcal{P}$. Choose an integer n such that $h^n(f^{-1}(\mathcal{P})$ and $\pi(\mathcal{T}_{(w,v)})$ are distance k a part, and, moreover, we may choose such an n so that $h^n(f^{-1}(\mathcal{P})) \subseteq \mathcal{T}_{(v,w)}$. Then, by replacing $\mathcal{P}$ and h by $h^n(f^{-1}(\mathcal{P}))$ and $h^n f^{-1} h f h^{-n}$, we may assume that $\mathcal{P}$ and $\mathcal{P}^{k-1}$ are contained in $\mathcal{T}_{(v,w)}$. If $K = \mathrm{Fix}_{F_{k,e}}(\mathcal{P}^{k-1})$, then $K \subseteq G^{+_k}$, and by the previous lemma, $K = [h, K] \subseteq H$ since H is normalised by G^{+_k}. Since $\mathcal{P}^{k-1} \subseteq \mathcal{T}_{(v,w)}$, $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)}) \subseteq K \subseteq H$ which completes the proof. □

3.4.4. A k-closures analogue of Universal groups. Motivated by this work on k-closures by Banks-Elder-Willis, in [**Tor20**], among other work, Tornier defines a generalised notion of universal groups where the local actions are defined on balls of radius k around each vertex instead of on balls of radius one like in the standard case of universal groups. These groups, denoted by $U_k(F)$, share a number of similar properties with the universal groups discussed earlier in the chapter and also satisfy Property P_k which will be useful to us later in the article. First we give the precise definition of these groups and we will follow similar notation to that used in [**Tor20**]:

Consider the regular tree $\mathcal{T}_d$ with its legal labelling $\lambda : E\mathcal{T}_d \to \Omega$ as discussed earlier in the case of universal groups. Denote by $B_{d,k}$ a tree isomorphic to a ball of radius k in $\mathcal{T}_d$ with the corresponding legal labelling. Then there exists, for each vertex $v \in V\mathcal{T}_d$, a unique label preserving isomorphism $\lambda_{v,k} : B(v,k) \to B_{d,k}$ from $B(v,k) \subseteq \mathcal{T}_d$ onto $B_{d,k}$. For each automorphism $\alpha \in \mathrm{Aut}(\mathcal{T}_d)$, its k-local action at a vertex $v \in V\mathcal{T}_d$ is then defined as:

$$\sigma_k(\alpha, v) := \lambda_{\alpha(v),k} \circ \alpha \circ \lambda_{v,k}^{-1}$$

which can easily be seen to be an element of $\mathrm{Aut}(B_{d,k})$. The k-closure analogue of the universal groups is then defined in the obvious way as:

Definition 3.30. Let $F \leqslant \mathrm{Aut}(B_{d,k})$. The universal group $U_k(F)$ is defined as $U_k(F) := \{\alpha \in \mathrm{Aut}(B_{d,k}) \mid \sigma_k(\alpha, v) \in F \text{ for all } v \in V\mathcal{T}_d\}$.

We remark that in the case when $k = 1$, we just have the usual definition of universal groups. Similar to the standard universal groups, the groups $U_k(F)$ satisfy the following properties:

Proposition 3.31. *Fix $k \in \mathbb{N}$ and let $F \leqslant \mathrm{Aut}(B_{d,k})$. The following properties hold:*

(i) $U_k(F)$ is closed in $\mathrm{Aut}(\mathcal{T}_d)$
(ii) $U_k(F)$ is vertex-transitive
(iii) $U_k(F)$ is compactly generated.

PROOF. The proof of parts (i) and (ii) is identical to the proof for universal groups that we saw earlier. For (iii), first note that $U_1(\{\mathrm{id}\}) \subseteq U_k(F)$. From the proof of Proposition 3.2(v), recall that there exists $\alpha_1, \alpha_2, \ldots, \alpha_d \in U_1(\{\mathrm{id}\}) \subseteq$

$U_k(F)$ such that $U_1(\{\mathrm{id}\}) \cong \langle\alpha_1\rangle * \langle\alpha_2\rangle * \cdots * \langle\alpha_d\rangle$. Then following a similar argument to the proof of Proposition 3.2(v), it can be shown that $U_k(F)_v \cup \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ is a compact generating set for $U_k(F)$. □

In particular, from the proposition, we see that the groups $U_k(F)$ are compactly generated, totally disconnected locally compact groups. It is also worth noting for use later that these groups also satisfy Property P_k:

Proposition 3.32. *Let $F \leqslant \mathrm{Aut}(B_{d,k})$ and fix $k \in \mathbb{N}$. The group $U_k(F)$ satisfies Property P_k.*

Proof. Since $U_k(F)$ is closed, by Proposition 3.23, we just need to show that $U_k(F)$ satisfies Property IP_k. So let $e = \{v, w\} \in E\mathcal{T}_d$ and $F_{k,e} = \mathrm{Fix}_{U_k(F)}(B(v,k) \cap B(w,k))$. It is easy to see that $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)}) \subseteq \mathrm{Fix}_{U_k(F)}(B(v,k) \cap B(w,k))$ so we just need to show that $\mathrm{Fix}_{U_k(F)}(B(v,k) \cap B(w,k)) \subseteq \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$. To this end, let $g \in \mathrm{Fix}_{U_k(F)}(B(v,k) \cap B(w,k))$. Define an automorphism g_1 so that $\sigma_k(g_1, v) = \sigma_k(g, v)$ for $v \in V\mathcal{T}_{(v,w)}$ and $\sigma_k(g_1, v) = \mathrm{id}$ for $v \in V\mathcal{T}_{(w,v)}$. Similarly, define g_2 such that $\sigma_k(g_1, v) = \sigma_k(g, v)$ for $v \in V\mathcal{T}_{(w,v)}$ and $\sigma_k(g_1, v) = \mathrm{id}$ for $v \in V\mathcal{T}_{(v,w)}$. It can then be seen that $g_1 \in \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$ and $g_2 \in \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})$ and $g = g_1 g_2$. Thus it follows that $g \in \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$ which completes the proof. □

Using the simplicity theorem given earlier, the following result is then deduced:

Corollary 3.33. *Let $F \leqslant \mathrm{Aut}(B_{d,k})$ and fix $k \in \mathbb{N}$. The group $U_k(F)^{+_k}$ is simple.*

3.5. Almost Automorphism Groups of Trees

Almost automorphism groups of trees, first studied by Neretin in [**Ner84**], and commonly referred to as *Neretin's groups*, are another novel example of groups of automorphisms acting on infinite trees. Specifically, given a regular rooted tree without leaves, the boundary of the tree, which can be identified with all the infinite rays starting at the root, forms a metric space under the choice of a suitable metric. This metric space is in fact a compact ultrametric space. The almost automorphism group of this tree can be recognised as a group of homeomorphisms of the boundary of the rooted tree, and these homeomorphisms are occasionally referred to as 'spheromorphisms' of the tree.

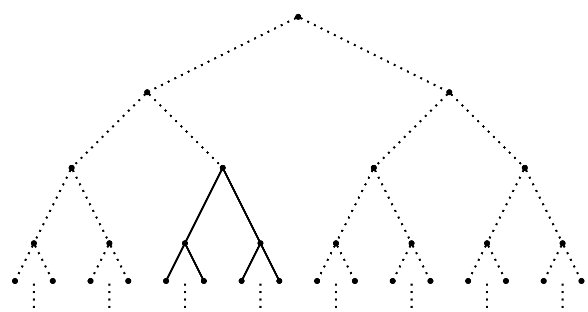
Almost automorphism groups of regular rooted trees enjoy many similar properties to the universal groups and Le Boudec's groups already discussed in this chapter: they are compactly generated, totally disconnected, locally compact groups. These groups also have the added properties of being non-discrete and abstractly simple, which was not always the case in the examples of universal groups and Le Boudec's groups. In the following, we give the reader an introduction to almost automorphism groups of trees and discuss some of their basic properties in more detail.

3.5.1. Metric Space Structure of the Boundary of a Rooted Tree. Here we comment on the metric space structure of the boundary of a rooted tree. Let $\mathcal{T}_{d,k}$ denote the rooted tree such that the root has degree k, and every other vertex has degree $d+1$. As discussed in the preliminaries section, the *boundary* of a tree is the collection of all equivalence classes of infinite rays, where two rays are considered equivalent if their intersection is also an infinite ray. In the case of the rooted tree $\mathcal{T}_{d,k}$, the boundary $\partial\mathcal{T}_{d,k}$ can be naturally identified with the collection of all infinite rays starting at the root vertex in $\mathcal{T}_{d,k}$. Given any ray $\xi \in \partial\mathcal{T}_{d,k}$, we will write $\xi = (v_i)_{i=1}^{\infty}$ where the v_i are the vertices on the ray ξ and v_1 is the root vertex. Then let ξ_n denote the path $(v_i)_{i=1}^{n}$ in $\mathcal{T}_{d,k}$. Given two rays $\xi, \xi' \in \partial\mathcal{T}_{d,k}$, define $\epsilon(\xi,\xi') = \sup\{n \in \mathbb{N} \mid \xi_n = \xi'_n\}$. This leads to the definition of a metric d on $\partial\mathcal{T}_{d,k}$ by

$$d(\xi,\xi') = e^{-\epsilon(\xi,\xi')}$$

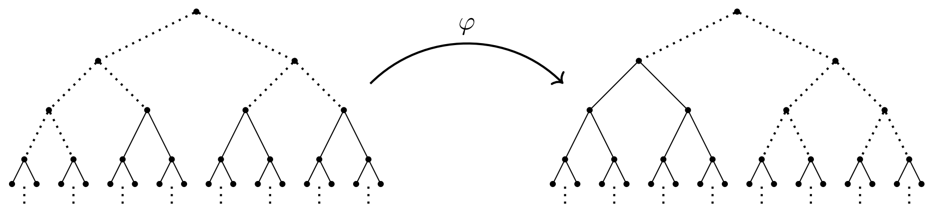
which is often called the *visual metric*. We use the convention that if $\epsilon(\xi,\xi') = \infty$, then $d(\xi,\xi') = 0$. It is easy to see that $\epsilon(\xi,\zeta) \geqslant \min\{\epsilon(\xi,\lambda),\epsilon(\lambda,\zeta)\}$, for $\xi,\zeta,\lambda \in \partial\mathcal{T}_{d,k}$; it follows that $d(\xi,\zeta) \leqslant \max\{d(\xi,\lambda), d(\lambda,\zeta)\}$ and hence $\partial\mathcal{T}_{d,k}$ is an ultrametric space. It can be shown that $(\partial\mathcal{T}_{d,k}, d)$ is in fact a compact ultrametric space and homeomorphic to the cantor set. It is easy to see why this is the case for the binary rooted tree: the boundary can be identified with the collection of all infinite binary strings, which is often taken as the definition of the cantor set.

Before moving on to defining almost automorphism groups of trees, we note that given $\xi = (v_i)_{i=1}^{\infty} \in \partial\mathcal{T}_{d,k}$, the ball $B(\xi, e^{-n})$ is precisely $\partial\mathcal{T}_{d,k}^{v_n}$, where $\mathcal{T}_{d,k}^{v_n}$ denotes the rooted subtree hanging below v_n in $\mathcal{T}_{d,k}$. Indeed, $B(\xi, e^{-n})$ is precisely all the infinite rays in $\partial\mathcal{T}_{d,k}$ that agree with ξ on the vertices $v_1, \ldots, v_n$ which can be naturally identified with $\partial\mathcal{T}_{d,k}^{v_n}$. Pictured below is a ball in the rooted tree $\mathcal{T}_{2,2}$.



3.5.2. The Almost Automorphism Group of $\mathcal{T}_{d,k}$. We will give two different definitions for an almost automorphism of the tree $\mathcal{T}_{d,k}$. The first definition provides a better visual picture and intuition than the second. That being said, the second definition given here is much more refined and is seen in many papers throughout the literature.

Given a finite subtree T of $\mathcal{T}_{d,k}$, we call T *complete* if for every vertex $v \in VT$ that is not a leaf, all the vertices adjacent to v in $\mathcal{T}_{d,k}$ are also contained in T. Following the paper [**Led19**], given two finite complete subtrees T_1 and T_2 of $\mathcal{T}_{d,k}$, an *honest almost automorphism* of $\mathcal{T}_{d,k}$ is a forest isomorphism $\varphi : \mathcal{T}_{d,k} \backslash T_1 \to \mathcal{T}_{d,k} \backslash T_2$. Then define an equivalence relation on the collection of all honest almost automorphisms as follows: let T_1, T_2, T_1', T_2' be finite complete subtrees of $\mathcal{T}_{d,k}$ and $\varphi : \mathcal{T}_{d,k} \backslash T_1 \to \mathcal{T}_{d,k} \backslash T_2$ and $\psi : \mathcal{T}_{d,k} \backslash T_1' \to \mathcal{T}_{d,k} \backslash T_2'$ two honest almost automorphisms. Define an equivalence relation by $\varphi \sim \psi$ if and only if there exists a finite complete subtree $T \subseteq \mathcal{T}_{d,k}$ containing $T_1 \cup T_1'$ such that $\varphi|_{\mathcal{T}_{d,k} \backslash T} = \psi|_{\mathcal{T}_{d,k} \backslash T}$.



An honest almost automorphism

Definition 3.34. An *almost automorphism* of $\mathcal{T}_{d,k}$ is an equivalence class of honest almost automorphisms under the equivalence relation $\sim$.

The almost automorphism group $\mathrm{AAut}(\mathcal{T}_{d,k})$ is defined as the collection of all almost automorphisms of $\mathcal{T}_{d,k}$. Essentially an almost automorphism permutes subtrees of $\mathcal{T}_{d,k}$, and two almost automorphisms are considered equivalent if they have the same action on the boundary of $\mathcal{T}_{d,k}$.

Composition of almost automorphisms is defined as follows: given two almost automorphisms $[\varphi]$ and $[\psi]$, choose representatives of the equivalences classes, say, $\tilde{\varphi} : \mathcal{T}_{d,k}\backslash T_1 \to \mathcal{T}_{d,k}\backslash T_2$ and $\tilde{\psi} : \mathcal{T}_{d,k}\backslash T_1' \to \mathcal{T}_{d,k}\backslash T_2'$ respectively. Let $T \subseteq \mathcal{T}_{d,k}$ be a finite complete subtree such that T contains $T_1 \cup T_2'$ and choose finite complete subtrees $T_3, T_4 \subseteq \mathcal{T}_{d,k}$ such that there exists honest almost automorphisms $\tilde{\varphi}' : \mathcal{T}_{d,k}\backslash T \to \mathcal{T}_{d,k}\backslash T_3$ and $\tilde{\psi}' : \mathcal{T}_{d,k}\backslash T_4 \to \mathcal{T}_{d,k}\backslash T$ which are also representatives of $[\varphi]$ and $[\psi]$ respectively. We may then compose these representatives and define $[\varphi] \circ [\psi] = [\tilde{\varphi}' \circ \tilde{\psi}']$.

We will now state the second, more refined definition. This definition can be found for instance in the paper by Le Boudec-Wesolek [**LBW19**]. Given two metric space (X, d_X) and (Y, d_Y), a map $\varphi : X \to Y$ is called a *homothety* if there exists a $C \in \mathbb{R}$ such that $d_X(x, x') = C d_Y(\varphi(x), \varphi(x'))$ for all $x, x' \in X$. An almost automorphism can then also be defined as follows:

Definition 3.35. An *almost automorphism* of $\mathcal{T}_{d,k}$ is a homeomorphism $\varphi \in \mathrm{Homeo}(\partial\mathcal{T}_{d,k})$ such that there exists a partition of $\partial\mathcal{T}_{d,k} = \bigsqcup_{i=1}^{n} B_i$, where B_i is a ball in $\partial\mathcal{T}_{d,k}$ for each i, and φ is a homothety when restricted to each of the B_i.

When $\mathcal{T}_{d,k}$ is a regular tree, the groups $\mathrm{AAut}(\mathcal{T}_{d,k})$ are commonly referred to as *Neretin's* groups, as these groups were first studied by Neretin in [**Ner84**]. To understand the topology on the groups $\mathrm{AAut}(\mathcal{T}_{d,k})$, we refer back to Proposition 3.7. First note that the group $\mathrm{Aut}(\mathcal{T}_{d,k})$ can be identified as a subgroup of $\mathrm{AAut}(\mathcal{T}_{d,k})$. If we let $G = \mathrm{AAut}(\mathcal{T}_{d,k})$ and $H = \mathrm{Aut}(\mathcal{T}_{d,k})$, it can be shown that these groups satisfy the hypotheses of Proposition 3.7. Thus, the groups $\mathrm{AAut}(\mathcal{T}_{d,k})$ are given the unique group topology such that the inclusion map of $\mathrm{Aut}(\mathcal{T}_{d,k})$ into $\mathrm{AAut}(\mathcal{T}_{d,k})$ is continuous and open. The groups $\mathrm{AAut}(\mathcal{T}_{d,k})$ then become totally disconnected locally compact groups with this topology. It is further the case that these groups are compactly generated and non-discrete. In Kapoudjian's paper [**Kap99**], it is also shown that Neretin's groups are always abstractly simple. We summarise these comments in the following theorem:

THEOREM 3.36. *The groups* $\mathrm{AAut}(\mathcal{T}_{d,k})$ *are (abstractly) simple, compactly generated, non-discrete, totally disconnected, locally compact groups.*

CHAPTER 4

Two Properties of Totally Disconnected Locally Compact Groups

4.1. Cartan-like Decompositions

In the theory of Lie groups and algebraic groups, studying decompositions of the groups is a useful tool in understanding the structure of the groups. Decompositions allow one to break the group down into smaller subsets, and by understanding the structure of the smaller, often easier to understand subsets, one can infer structural information about the whole group. The Cartan decomposition is a particularly well known decomposition studied in Lie theory. Given a group G and a compact subgroup $K \leqslant G$, a *Cartan decomposition* of G with respect to K is a double coset decomposition of the form:

$$G = \bigsqcup_{a \in A} KaK$$

where $A \subseteq G$ is a set of coset representatives.

In the theory of totally disconnected locally compact groups, some recent work has involved understanding whether results about Lie groups and algebraic groups transfer across to totally disconnected locally compact groups. Following this idea, in the paper [**CW20**], it was shown among other results, that the automorphism group of a label-regular tree admits a Cartan-like decomposition, and as a result of this, every continuous homomorphism from the simple subgroup $\mathrm{Aut}^+(\mathcal{T}_{\mathbf{a}})$ has closed range. Both these properties are shared in common with simple Lie groups. This work then leads to defining two properties, the *contraction group property* and the *closed range property*, which will be discussed shortly. The closed range property has an intimate connection with the contraction group property and Cartan-like decompositions of groups. We first recall some of the work from this paper, in particular, the result regarding Cartan-like decompositions of automorphism groups of label-regular trees:

THEOREM 4.1. [**CW20**, Theorem 3.1] *Let $\mathcal{T}_{\boldsymbol{a}}$ be a label-regular tree with labels in a set, Ω, and let $K = \mathrm{Aut}(\mathcal{T}_{\boldsymbol{a}})_v$ for a fixed vertex $v \in V\mathcal{T}_{\boldsymbol{a}}$. Let A be the set of all finite sequences in Ω that are compatible with $\mathcal{T}_{\boldsymbol{a}}$ and begin and end with the label $\lambda(v)$. For each $\alpha \in A$, choose $v_\alpha \in V\mathcal{T}_{\boldsymbol{a}}$ and $g_\alpha \in \mathrm{Aut}(\mathcal{T}_{\boldsymbol{a}})$ such that the sequence of labels of vertices on the unique path from v to v_α is α and $g_\alpha(v) = v_\alpha$. Then the double cosets $Kg_\alpha K$, $\alpha \in A$, are pairwise disjoint and*

$$\mathrm{Aut}(\mathcal{T}_{\boldsymbol{a}}) = \bigsqcup_{\alpha \in A} Kg_\alpha K.$$

Now, given a group G and a sequence $(g_i)_{i\in I} \subseteq G$, we define the *contraction group* of the sequence, denoted $\mathsf{con}((g_i)_{i\in I})$, to be $\mathsf{con}((g_i)_{i\in I}) := \{x \in G \mid g_i x g_i^{-1} \to \mathrm{id}_G\}$. It was shown in [**CW20**], that if we take any infinite subset of coset representative in the above Cartan-like decomposition of $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$, then either the subset is bounded or has a subsequence with non-trivial contraction group. This result led to the fact that any continuous homomorphism from the simple subgroup $\mathrm{Aut}^+(\mathcal{T}_{\mathbf{a}}) \leqslant \mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$, has closed range. In the remainder of this article, we will be aiming to further understand these properties in a broader context than what was studied in [**CW20**]. From now on, we will agree to use the following terminology for these properties:

Definition 4.2 (Contraction Group Property)**.** Let G be a topological group, $K \leqslant G$ a compact subgroup, and $A \subseteq G$ such that G admits the Cartan-like decomposition $G = \bigsqcup_{a\in A} KaK$. We say that this decomposition has the *contraction group property* if every sequence of elements in A is either bounded or has a subsequence with non-trivial contraction group. We will say that a group G has the contraction group property if it admits a Cartan-like decomposition satisfying the contraction group property.

Similarly, we define the closed range property as follows:

Definition 4.3 (Closed Range Property)**.** A topological group G is said to have the *closed range property* if every continuous homomorphism $\varphi : G \to G'$, for an arbitrary topological group G', has closed range i.e. $\varphi(G)$ is closed in G'.

Later it will be shown that the contraction group property for a group G does not depend on the choice of compact open subgroup K, however, it is not true in general that the contraction group property is independent of the choice of coset representatives. This will be discussed in more detail later. In the main theorem of [**CW20**], it was essentially shown that any simple group satisfying the contraction group property also has the closed range property. This is a generalisation of Theorem 4.1 in [**CW20**] and will be proved later in this chapter as well. This idea will be utilised to prove some more general closed range results for a larger class of t.d.l.c. groups acting on trees that were discussed in the prior chapter.

We now proceed to extend this work by developing some more general results concerning the contraction group and closed range properties.

4.2. The Contraction Group Property

In this section we will prove some more general results concerning the contraction group property. First we show that the contraction group property does not depend on the choice of compact open subgroup:

Proposition 4.4. *The contraction group property does not depend on the choice of compact open subgroup in the Cartan-like decomposition.*

Proof. Let G be a topological group and suppose that there exists a compact open subgroup $K \leqslant G$, and a set of coset representatives $A \subseteq G$ such that G admits a Cartan-like decomposition $G = \bigsqcup_{a\in A} KaK$ satisfying the contraction group property. Let $K' \leqslant K$ be another compact open subgroup of G. By compactness of K, there exists elements $g_1, \ldots, g_n \in G$ such that $K = \bigsqcup_{i=1}^{n} g_i K'$. Then G also decomposes as $G = \bigsqcup_{a\in A} \bigsqcup_{i,j} K'(g_i^{-1} a g_j)K'$. Now suppose we have a sequence of coset representatives $(h_k)_{k=1}^{\infty} \subseteq \{g_i^{-1} a g_j\}_{a\in A, i,j\in\{1,2,\ldots,n\}}$. We need to show that the sequence $(h_i)_{i=1}^{\infty}$ is either bounded or has a subsequence with non-trivial contraction group.

If the sequence is bounded, we are done, so we may suppose that it is unbounded and show that it has a subsequence with non-trivial contraction group. Since there is only finitely many g_i, by passing to a subsequence of the h_i, we may suppose that for each i, $h_i = g_l^{-1} a_i g_m$ for some fixed natural numbers l and m, and $a_i \in A$ for each i. Now, by assumption, there exists a subsequence $(a_{i_j})_{j=1}^{\infty} \subseteq (a_i)_{i=1}^{\infty}$ and a non-trivial $x \in \mathsf{con}((a_{i_j})_{j=1}^{\infty})$. It is then easy to compute that $\tilde{x} = g_m^{-1} x g_m$ is in the contraction group of the subsequence $(h_{i_j})_{j=1}^{\infty}$ and non-trivial. Thus the decomposition $G = \bigsqcup_{a \in A} \bigsqcup_{i,j} K'(g_i^{-1} a g_j) K'$ has the contraction group property.

Conversely, retaining the notation from above, suppose that K'' is a compact open subgroup of G containing K. Then there exists a subset $A'' \subseteq A$ such that $G = \bigsqcup_{a \in A''} K'' a K''$. Since every sequence in A is either bounded or has a subsequence with non-trivial contraction group, the same property holds for the set A''. Thus the decomposition $G = \bigsqcup_{a \in A''} K'' a K''$ has the contraction group property.

We have shown that if a compact open subgroup of G contains K or is contained in K, then G admits a Cartan-like decomposition with respect to this compact open subgroup satisfying the contraction group property. Now suppose that L is an arbitrary compact open subgroup not necessarily contained in or containing K. Then $L \cap K$ is a compact open subgroup contained in K, hence G admits a Cartan-like decomposition satisfying the contraction group property with respect to $L \cap K$. Then it follows that G also admits a Cartan-like decomposition with respect to L satisfying the contraction group property since L contains $L \cap K$ and the contraction group property holds for the decomposition with respect to L. □

Ideally, the contraction group property would also not depend on the choice of coset representatives in the Cartan-like decomposition, however, unfortunately, this is not always the case as shown in the following example:

Example 4.5. Take the group $G = \mathrm{PGL}(\mathbb{Q}_p)$ and the compact open subgroup $K = \mathrm{PGL}(\mathbb{Z}_p)$. G admits a Cartan-like decomposition:

$$G = \bigsqcup_{n \in \mathbb{N}} K g^n K$$

where $g = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It can be checked that the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is a non-trivial element in the contraction group of the sequence $(g^n)_{n=1}^{\infty}$. Then since $y_n = \left(\begin{smallmatrix} 1 & 0 \\ p^{n/3} & 1 \end{smallmatrix}\right)$ (where we assume we take the ceiling of $n/3$ when it is non-integral) is in the compact open subgroup K for each $n \in \mathbb{N}$, the elements $g_n = g^n y_n = \left(\begin{smallmatrix} p^n & 0 \\ p^{n/3} & 1 \end{smallmatrix}\right)$ also form a set of coset representatives for a Cartan-like decomposition for $\mathrm{PGL}(\mathbb{Q}_p)$. We claim that the contraction group of every subsequence of the sequence $(g_n)_{n=1}^{\infty}$ is trivial.

Suppose that $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}(\mathbb{Q}_p)$ and is in the contraction group of the sequence $(g_n)_{n=1}^{\infty}$. Then, it may be checked that:

$$\tilde{g}_n = g_n h g_n^{-1} = \begin{pmatrix} a - bp^n & bp^n \\ (a-d)p^{-2n/3} + cp^{-n} - bp^{-n/3} & bp^{n/3} + d \end{pmatrix}$$

Since it is assumed that $\tilde{g}_n \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we must have $a = d = 1$. Then,

$$\tilde{g}_n = \begin{pmatrix} 1 - bp^n & bp^n \\ cp^{-n} - bp^{-n/3} & 1 + bp^{n/3} \end{pmatrix}$$

It then follows that we must have $b = c = 0$ for $\tilde{g}_n$ to converge to the identity, otherwise, the norm of the bottom left entry of the matrix will diverge to ∞, and hence the bottom left entry must diverge. Thus h is the identity and so $\mathsf{con}((g_n)_{n=1}^{\infty})$ is trivial. It is clear that every subsequence of the sequence $(g_n)_{n=1}^{\infty}$ must also have trivial contraction group. This demonstrates that if one Cartan-like decomposition of a group has the contraction group property, then another decomposition may not, even if we keep the same compact subgroup K.

Although, in general, the choice of coset representatives in a Cartan-like decomposition can effect whether the decomposition will have the contraction group property or not, for some classes of groups this is not the case:

Proposition 4.6. *Let $\mathcal{T}_{\mathbf{a}}$ be a label regular tree. The contraction group property for $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$ does not depend on the choice of coset representatives.*

Proof. Assume the contraction group property for coset representatives holds for some decomposition $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}}) = \bigsqcup_{\alpha \in A} K g_\alpha K$, where $K = \mathrm{Aut}(\mathcal{T}_{\mathbf{a}})_v$ is some compact open subgroup with $v \in V\mathcal{T}_{\mathbf{a}}$, and $\mathcal{A} = \{g_\alpha \mid \alpha \in A\}$ is a set of coset representatives for the decomposition and A an indexing set. We may assume that A is in one-to-one correspondence with the set of finite sequence of labels compatible with $\mathcal{T}_{\mathbf{a}}$ as discussed in [**CW20**].

Suppose there exists another set of coset representatives $\{h_\beta \mid \beta \in B\}$ such that $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}}) = \bigsqcup_{\beta \in B} K h_\beta K$, where B is some indexing set and each of the h_β are automorphisms in $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$. We need to show that this new decomposition satisfies the contraction group property. Let $\{h_{\beta_i}\}_{i=1}^{\infty}$ be a subsequence of the coset representatives. Since each of the h_{β_i} are in $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$ and $\mathrm{Aut}(\mathcal{T}_{\mathbf{a}}) = \bigsqcup_{\alpha \in A} K g_\alpha K$, for each i we may write $h_{\beta_i} = k_i g_{\alpha_i} k_i'$ for some $k_i, k_i' \in K$ and $g_{\alpha_i} \in \mathcal{A}$. If the sequence $(g_{\alpha_i})_{i=1}^{\infty}$ is bounded then the sequence $(h_{\beta_i})_{i=1}^{\infty}$ is also since each of the k_i and k_i' are contained in the compact open subgroup K. So we assume that the sequence $(g_{\alpha_i})_{i=1}^{\infty}$ is unbounded and show that there is a non-trivial element in the contraction group of some subsequence of the h_{β_i}.

The remainder of the proof follows a similar argument to the proof of [**CW20**, Theorem 4.1]. Since the sequence $(g_{\alpha_i})_{i=1}^{\infty}$ is unbounded, by passing to a subsequence if necessary, we may suppose $d(v, g_{\alpha_i}(v)) \geqslant i$ for each i, and since $\mathcal{T}_{\mathbf{a}}$ is locally finite, by passing to a subsequence if necessary, we may suppose that the first vertex on the path from v to $g_{\alpha_i}(v)$ is always $w \in V\mathcal{T}_{\mathbf{a}}$. Then since k_i and k_i' fix v for each i, we have that $d(v, h_{\beta_i}) \geqslant i$ for each i. Also, since $\mathcal{T}_{\mathbf{a}}$ is locally finite, we may suppose by passing to subsequences if necessary, that each of the k_i and k_i' have the same action on $B(v, 1)$ in $\mathcal{T}_{\mathbf{a}}$. Then it follows from this that the first vertex on the path from v to $h_{\beta_i}(v)$ always passes through a fixed vertex say $w' \in V\mathcal{T}_{\mathbf{a}}$.

Now, if infinitely many of the h_{β_i} are translations with v on the axis, then by passing to subsequence is necessary, we may suppose that they all are. Choosing an $x \in \mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$ that fixes $\mathcal{T}_{(v,w')}$ and acts non-trivially on $\mathcal{T}_{(w',v)}$ gives a non-trivial element in the contraction group of the sequence $(h_{\beta_i})_{i=1}^{\infty}$. If only finitely many of the h_{β_i} are translations with v on the axis, again, by passing to a subsequence, we may suppose that each of the h_{β_i} are either elliptic elements, inversions or translations with v not on the axis. Choosing $x \in \mathrm{Aut}(\mathcal{T}_{\mathbf{a}})$ that acts non-trivially on $\mathcal{T}_{(v,w')}$ and fixes $\mathcal{T}_{(w',v)}$ gives a non-trivial element in the contraction group of the sequence $(h_{\beta_i})_{i=1}^{\infty}$. □

4.3. The Closed Range Property

This section will study some more general results concerning the closed range property in topological groups. First we prove a generalisation of Theorem 4.1 from [**CW20**] that will later be applied to groups acting on trees to develop some new closed range properties for certain groups. We also prove a result regarding when the closed range property is passed to supergroups.

THEOREM 4.7. *Let G be a topologically simple topological group that has the contraction group property. Then G has the closed range property.*

PROOF. Suppose the hypotheses of the theorem. Let $K \leqslant G$ be a compact open subgroup and $A \subseteq G$ such that G admits a Cartan-like decomposition $G = \bigsqcup_{a \in A} KaK$ with the contraction group property, and suppose that $\varphi : G \to H$ is a non-trivial continuous homomorphism to some topological group H. Consider a sequence $(g_i)_{i=1}^{\infty} \subseteq G$ and suppose that $\varphi(g_i)$ converges to $h \in H$. It must be shown that $h \in \varphi(G)$. Now, there are sequences $(k_i)_{i=1}^{\infty}$, $(k_i')_{i=1}^{\infty} \subseteq K$ and $(a_i)_{i=1}^{\infty} \subseteq A$ such that $g_i = k_i a_i k_i'$ for each i. Passing to a subsequence if necessary, we may suppose, by compactness of K, that the sequences $(k_i)_{i=1}^{\infty}$ and $(k_i')_{i=1}^{\infty}$ converge to elements $k, k' \in K$ respectively. Then $\varphi(a_i) = \varphi(k_i)^{-1}\varphi(g_i)\varphi(k_i')^{-1} \to \varphi(k)^{-1} h \varphi(k')^{-1}$ as $i \to \infty$. Thus the sequence $(\varphi(a_i))_{i=1}^{\infty}$ converges.

If the sequence $(a_i)_{i=1}^{\infty}$ is bounded, it may be supposed, by passing to a subsequence if necessary, that the sequence is constant. Then $a_i = a \in A$ for each i and $h = \varphi(k)\varphi(a)\varphi(k') \in \varphi(G)$. Thus the proof is complete. So suppose that the sequence $(a_i)_{i=1}^{\infty}$ is unbounded and set $\hat{a} := \lim_{i\to\infty} \varphi(a_i)$. By assumption, there exists a subsequence $(a_{i_j})_{j=1}^{\infty} \subseteq (a_i)_{i=1}^{\infty}$ and a non-trivial $x \in \mathsf{con}((a_{i_j})_{j=1}^{\infty})$. Then $\hat{a}\varphi(x)\hat{a}^{-1} = \lim_{i\to\infty} \varphi(a_i x a_i^{-1}) = \lim_{j\to\infty} \varphi(a_{i_j} x a_{i_j}^{-1}) = \mathrm{id}_H$ by definition of the contraction subgroup for the sequence $(a_{i_j})_{j=1}^{\infty}$ and continuity of φ. Hence the kernel of φ contains $\mathsf{con}((a_{i_j})_{i=1}^{\infty})$ and so φ must be the trivial homomorphism because G is topologically simple, a contradiction. This completes the proof. □

We now move onto showing that, under certain assumption, the closed range property is passed to supergroups, in particular, we show that if a locally compact group

G has a cocompact subgroup with the closed range property, then G also has the closed range property. We need the following two lemmas for the proof:

Lemma 4.8. *Let G be a topological group and $A, B \subseteq G$ with A compact and B closed. Then the set AB is closed.*

PROOF. Let $(c_i)_{i \in I}$ be a net in $C = AB$ with I some indexing set. Suppose that $c_i \to c$ for some $c \in G$. We will show that $c \in C$. For each $i \in I$, choose $a_i \in A$ and $b_i \in B$ such that $c_i = a_i b_i$. By compactness of A, there exists a subnet $(a_j)_{j \in J} \subseteq (a_i)_{i \in I}$ ($J \subseteq I$), and an $a \in A$ such that $a_j \to a$. Then it follows by continuity of the group operations in G, and closedness of B, that $b_j = a_j^{-1} c_j \to b$ for some $b \in B$. Thus $c_j \to ab$ and hence $c_i \to ab$. Since $ab \in AB$, it follows that AB is closed. □

Recall that given a topological group G and a subgroup $H \leqslant G$, H is said to be *cocompact* in G if its quotient G/H is compact in the quotient topology. We prove the following fact about locally compact groups:

Lemma 4.9. *Let G be a locally compact group and suppose that H is a cocompact subgroup of G. Then there exists a compact set $K \subseteq G$ such that $G = KH$.*

PROOF. Let $\mathcal{U}$ be the collection of open sets in G with compact closure and let $\pi : G \to G/H$ the canonical map. Then, since π is an open map, $\{\pi(U) | U \in \mathcal{U}\}$ forms an open covering of G/H. By compactness of G/H, there is a finite subcover say $\pi(U_1), \ldots, \pi(U_n)$ of G/H. Then it follows that $K = \bigcup_{i=1}^{n} \overline{U_{\alpha_i}}$ is a compact subset of G, being a finite union of compact sets, and $\pi(K) = KH = G$ by construction. □

Proposition 4.10. *Let G be a locally compact group. Suppose that H is a cocompact subgroup of G with the closed range property. Then G also has the closed range property.*

PROOF. Let G be a locally compact group and suppose that H is a cocompact subgroup of G with the closed range property. Let K be a compact subset of G such that $G = KH$, which exists by the previous lemma. Let $\varphi : G \to \hat{G}$ be a continuous homomorphism to an arbitrary topological group $\hat{G}$. By assumption, we know that $\varphi(H)$ is closed in $\hat{G}$, since H satisfies the closed range property and the restriction of φ to H is continuous. Since φ is continuous, $\varphi(K)$ is compact in $\hat{G}$, and then by Lemma 4.8, $\varphi(G) = \varphi(K)\varphi(H)$ is closed being the product of a compact set and a closed set. □

CHAPTER 5

The Contraction Group and Closed Range Properties in Tree Automorphism Groups

In this chapter we study the contraction group and closed range properties in some of the automorphism groups of trees seen in Chapter 3. In the first section, we provide an example of how the contraction group property can fail for a Cartan-like decomposition of the Le Boudec groups $G(F, F')$. We then proceed to develop some closed range results for the simple subgroups G^{+_k} encountered in Chapter 3 as well. This leads to some closed range results for the (generalised) universal groups and more generally groups acting on trees with a locally semiprimitive action. The section ends with some comments on almost automorphism groups and the relation between commensurated subgroups and the closed range property.

5.1. Le Boudec's Groups

We will now briefly discuss the contraction group and closed range properties in the context of Le Boudec's groups. It can be easily deduced from Proposition 3.10 that the groups $G(F, F')$ do not in general satisfy the closed range property: when $G(F, F') \neq U(F')$, the inclusion map $G(F, F') \hookrightarrow \mathrm{Aut}(\mathcal{T}_d)$ is continuous but not closed. In the following we give an explicit example of a Cartan-like decomposition of one of Le Boudec's groups that does not satisfy the contraction group property. We show that there exists a Cartan-like decomposition of $G(\langle(123)\rangle)$ that does not satisfy the contraction group property. To do this, it is suffice to find such a decomposition for the subgroup $G(\langle(123)\rangle)_v$ for some fixed $v \in V\mathcal{T}_d$. Indeed, we can extend a set of coset representatives for a Cartan-like decomposition of $G(\langle(123)\rangle)_v$ to a set of coset representatives for a Cartan-like decomposition of $G(\langle(123)\rangle)$, and if the contraction group property is not satisfied for the decomposition of $G(\langle(123)\rangle)_v$, then it will not be satisfied for the decomposition of $G(\langle(123)\rangle)$ either.

Recall from the section on Le Boudec's groups in Chapter 2, that the topology on $G(\langle\langle(123)\rangle\rangle)$ is the unique group topology such that the inclusion map $U(\langle\langle(123)\rangle\rangle) \hookrightarrow G(\langle\langle(123)\rangle\rangle)$ is continuous and open. In particular, the compact open subgroups of $G(\langle\langle(123)\rangle\rangle)$ are precisely the compact open subgroups in $U(\langle\langle(123)\rangle\rangle)$, and their translations and finite unions in $G(\langle\langle(123)\rangle\rangle)$. Hence, for our Cartan-like decomposition of $G(\langle\langle(123)\rangle\rangle)_v$, we will take the compact open subgroup to be $K = U(\langle\langle(123)\rangle\rangle)_v$.

First, we state the following lemma which demonstrates a method for choosing coset representatives in a Cartan-like decomposition for the Le Boudec groups:

Lemma 5.1. *Let $F \leqslant F' \leqslant \mathrm{Sym}(d)$ be two permutation groups and $K = U(F)_v$ for some vertex $v \in V\mathcal{T}_d$. Set $A_n = \{g \in G(F,F')_v \backslash U(F)_v \mid S(g) \subseteq B(v,n)\}$ and $A = \bigcup_{i=1}^{\infty} A_i$. Define an equivalence relation on A by $a_1 \sim a_2$ if and only if there exists $k, k' \in K$ such that $a_1 = ka_2k'$ and let $\tilde{A}$ be a set of representatives of equivalences classes in $A/\sim$. Then each of the double-cosets KaK are pairwise disjoint for each $a \in \tilde{A}$ and*

$$G(F,F')_v = \bigsqcup_{a\in\tilde{A}} KaK$$

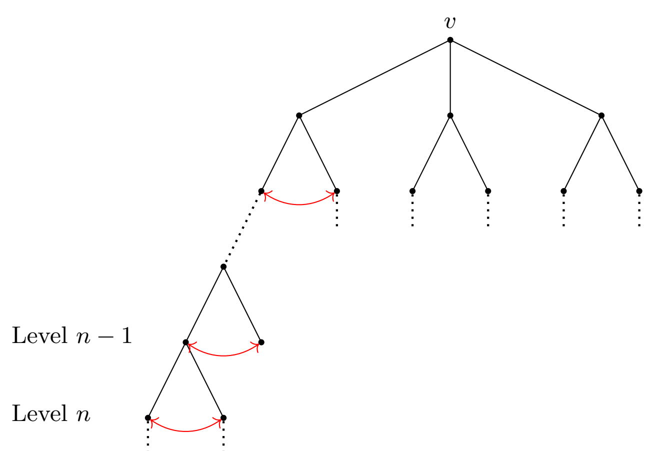
Proof. Let $g \in G(F,F')_v$ and suppose that $S(g) \subseteq B(v,n)$. Then there exists $a \in A_n$ such that $a^{-1}g \in U(F)_v$; for instance, take $a = g$ if $g \in G(F,F')_v \backslash U(F)_v$ or $a = \mathrm{id}$ if $g \in U(F)_v$. Then it follows that $g = ak$ for some $k \in K$ so $G(F,F') = \bigcup_{a\in A} aK = \bigcup_{a\in A} KaK$. Now, there exists a unique element $\tilde{a} \in \tilde{A}$ such that $\tilde{a} = kak'$ for some $k, k' \in K$. It follows that $g \in K\tilde{a}K$ and hence $G(F,F') = \bigcup_{a\in\tilde{A}} KaK$.

To show that the cosets are disjoint, suppose that $KaK \cap Ka'K \neq \varnothing$ for some distinct $a, a' \in \tilde{A}$. Then we must have that $k_1ak_1' = k_2a'k_2'$ for some $k_i, k_i' \in K$ $(i = 1, 2)$. It follows that $a = ka'k'$ for some $k, k' \in K$ i.e. $a \sim a'$. Hence we must have that $KaK = Ka'K$. Thus the cosets are either equal or disjoint and hence $G(F,F')_v = \bigsqcup_{a\in\tilde{A}} KaK$. □

Thus choosing coset representatives for a Cartan-like decomposition of the groups $G(F,F')_v$ is a matter of choosing one automorphism from each of the equivalence classes in $A/\sim$. In particular, to find a sequence of coset representatives that do not satisfy the contraction group property, it is suffice to find a sequence in $G(\langle\langle(123)\rangle\rangle)_v$

that has trivial contraction group, such that each of the elements are not equivalent under the equivalence relation $\sim$ described in the lemma. Then we may extend this sequence to a sequence of coset representatives for a Cartan-like decomposition of $G(\langle\langle(123)\rangle\rangle)_v$ that does not satisfy the contraction group property.

Thus, for each $n \in \mathbb{N}$, define $a_n \in G(\langle\langle(123)\rangle\rangle)_v$ to 'switch' the two left most vertices on each of the levels $2, \ldots, n$ of the 3-regular (pictured as the rooted tree $\mathcal{T}_{2,3}$) as illustrated below:



The automorphism a_n

Since $U(\langle\langle(123)\rangle\rangle)_v$ consists of only those automorphisms that cyclically permute the vertices on level 1 of the above tree (and have the same local action at each vertex), it is easy to see that a_n is not equivalent to a_m for any $m \neq n$. We just need to show that every subsequence of $(a_n)_{n=1}^{\infty}$ has trivial contraction group. This is indeed the case since $G(\langle\langle(123)\rangle\rangle)$ is a discrete group, however, we will also give the following argument which provides more intuition as to why the contraction group is trivial.

If $x \in \mathsf{con}((a_{n_i})_{i=1}^{\infty})$ for some subsequence $(a_{n_i})_{i=1}^{\infty} \subseteq (a_n)_{n=1}^{\infty}$, then x has to fix the root v otherwise $a_n x a_n^{-1}$ would shift v to a vertex distance $d(v, x(v))$ away from v for all n, hence $\{a_{n_i} x a_{n_i}^{-1}\}_{i=1}^{\infty}$ could not converge to the identity. Also, x clearly has to act trivially on the two right most branches of the above tree. If x acts non-trivially on the left most branch of the tree, then there is some level, say level

m, on which x switches (atleast) two vertices, and it is easy to see that $a_{n_i} x a_{n_i}^{-1}$ acts non-trivially on level m for each i. Thus x must be trivial and hence every subsequence of $(a_n)_{n=1}^{\infty}$ has trivial contraction group.

5.2. Closed Groups Acting on Trees

We now extend the work on Cartan-like decompositions and the closed range property seen in [**CW20**] to a larger class of groups acting on trees. We show in the following, that under standard assumptions, for any closed subgroup $G \leqslant \mathrm{Aut}(\mathcal{T})$, G^{+_k} has the closed range property. This will have consequences for universal groups, k-closures of groups acting on trees and non-discrete locally semi-primitive groups. Throughout this section we assume that $\mathcal{T}$ is an arbitrary infinite locally finite tree without leaves. First we note that closed subgroups of the automorphism group of a locally finite tree $\mathcal{T}$ admit a Cartan-like decomposition with a vertex stabiliser as the compact open subgroup:

Proposition 5.2. *Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ be a closed subgroup. For any $v \in V\mathcal{T}$, the vertex stabiliser $K = G_v \leqslant G$ is a compact open subgroup of G, and G admits a Cartan-like decomposition $G = \bigsqcup_{a \in A} KaK$ for some $A \subseteq G$. Moreover, A can be chosen so that there is exactly one element of A for each orbit of G acting on $Gv \cap S(v,n)$ for each $n \in \mathbb{N}$.*

Proof. Fix $v \in V\mathcal{T}$. It is clear that $K = G_v$ is a compact open subgroup in the subspace topology on G, being the intersection of the compact open subgroup $\mathrm{Aut}(\mathcal{T})_v$ with G. Now, let Gv denote the orbit of v under the action of G. To show that G admits a Cartan-like decomposition, enumerate the orbits of K acting on the spheres $S(v,n) \cap Gv \subseteq \mathcal{T}$ for each $n \in \mathbb{N}$. For each orbit, choose a vertex w in that orbit and an automorphism $a_w \in G$ that sends v to w. Let A be the collection of all the chosen a_w. We claim that $G = \bigsqcup_{a \in A} KaK$. Indeed, let $g \in G$. There exists a vertex $w \in V\mathcal{T}$ in the K-orbit of $g(v)$ and an automorphism $a_w \in A$ that sends v to w. Let $k \in K$ such that $kg(v) = w$. Then $a_w^{-1} kg(v) = v$, so $a_w^{-1} kg \in K$, and it follows that $g \in K a_w K$. If there exists $a_{w_1}, a_{w_2} \in A$, $a_{w_1} \neq a_{w_2}$, such that

$Ka_{w_1}K = Ka_{w_2}K$ then this would contradict that w_1 and w_2 are in different K-orbits. Thus the double cosets KaK, for $a \in A$, are disjoint for distinct a and hence it follows that $G = \bigsqcup_{a\in A} KaK$. □

In particular, for any $F \leqslant \mathrm{Sym}(d)$, $U(F)$ admits a Cartan-like decomposition $U(F) = \bigsqcup_{a\in A} KaK$ where $K = U(F)_v$ for a fixed $v \in V\mathcal{T}_d$ and the coset representatives are constructed as above. Recall from [**BM00**] that a group is called ∞-transitive if the stabiliser of a vertex v acts transitively on $S(v,n)$ for every n. If $U(F)$ is ∞-transitive, then $U(F)$ admits a Cartan-like decomposition whose coset representatives are powers of a single translation:

Corollary 5.3. *Let $F \leqslant \mathrm{Sym}(d)$ and assume that $U(F)$ is ∞-transitive. Let $\alpha \in U(F)$ be a translation and $v \in V\mathcal{T}_d$ be a vertex on the axis of α. Then $U(F) = \bigsqcup_{n\in\mathbb{Z}} K\alpha^n K$ where $K = U(F)_v$.*

Proof. Let $g \in U(F)$. If g fixes v, then clearly $g \in \bigsqcup_{n\in\mathbb{Z}} K\alpha^n K$. So we may suppose that $g(v) \neq v$. Since F is ∞-transitive, there exists a $k_1 \in K$ such that $k_1 g(v)$ is on the axis of α. Then there exists an integer m such that $\alpha^m k_1 g(v) = v$. Let $k_2 \in K$ such that $\alpha^m k_1 g = k_2$. It follows that $g = k_1^{-1}\alpha^{-m}k_2 \in \bigsqcup_{n\in\mathbb{Z}} K\alpha^n K$. The cosets are clearly disjoint. □

For use in the forthcoming theorem, we need the following two lemma's. The first lemma is just Lemma 4 in [**MV12**] restated for use here. We direct the reader to [**MV12**] for the proof.

Lemma 5.4. *Suppose that $G \leqslant \mathrm{Aut}(\mathcal{T})$ does not stabilise any non-empty subtree of $\mathcal{T}$. Then the following hold:*

(i) Suppose that there is some edge $\{u,v\} \in E\mathcal{T}$ such that the pointwise stabiliser of both the half-trees $\mathcal{T}_{(u,v)}$ and $\mathcal{T}_{(v,u)}$ are non-trivial. Then the pointwise stabiliser of every half-tree in $\mathcal{T}$ is non-trivial.

(ii) Suppose that there is some edge $\{u,v\} \in E\mathcal{T}$ such that the pointwise stabiliser of $\mathcal{T}_{(u,v)}$ is trivial while the pointwise stabiliser of $\mathcal{T}_{(v,u)}$ is non-trivial. Then G must fix an end of $\mathcal{T}$.

Now, we prove the following lemma which is a consequence of the previous lemma and Lemma 3.24:

Lemma 5.5. *Suppose that $G \leqslant \mathrm{Aut}(\mathcal{T})$ and assume that G does not stabilise any proper non-empty subtree or fix an end of $\mathcal{T}$, and satisfies Property P_k. If G^{+_k} is non-trivial, then the fixator in G^{+_k} of every half tree in $\mathcal{T}$ is non-trivial.*

Proof. Since G^{+_k} is normal in G, by Lemma 3.24, G^{+_k} does not stabilise any proper non-empty subtree or end of $\mathcal{T}$. Since G^{+_k} is non-trivial, there exists an edge $e = \{v, w\} \in E\mathcal{T}$ and a non-trivial element $g \in F_{k,e} = \mathrm{Fix}_G(B(v, k) \cap B(w, k))$. Now we know that $F_{k,e} = \mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$ since G satisfies Property P_k. Thus, since $F_{k,e}$ is non-trivial, there must exist a non-trivial element g' in either $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(w,v)})$ or $\mathrm{Fix}_{F_{k,e}}(\mathcal{T}_{(v,w)})$. Clearly $g' \in G^{+_k}$. Since G^{+_k} does not stabilise any non-empty subtree or fix an end of $\mathcal{T}$, an application of Lemma 5.4(ii) followed by an application of Lemma 5.4(i) then shows that the stabiliser in G^{+_k} of every half tree in $\mathcal{T}$ must be non-trivial. □

We now come to the following theorem which shows that for closed subgroups $G \leqslant \mathrm{Aut}(\mathcal{T})$, the groups G^{+_k} have the closed range property under certain assumptions:

Theorem 5.6. *Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ be a closed subgroup and suppose that G does not stabilise any proper non-empty subtree or fix an end of $\mathcal{T}$. If G satisfies Property P_k, then G^{+_k} has the closed range property.*

Proof. First we note that G^{+_k} is open in G since it contains for instance the open neighbourhood $\mathcal{U}(\mathrm{id}, B(v, k) \cap B(w, k))$ of the identity, where $\{v, w\} \in E\mathcal{T}$. Since G^{+_k} is open in G, it is also closed in G, and since G is closed in $\mathrm{Aut}(\mathcal{T})$, it follows that G^{+_k} is closed in $\mathrm{Aut}(\mathcal{T})$. By Proposition 5.2, G^{+_k} admits a Cartan-like decomposition $G = \bigsqcup_{a \in A} KaK$, with $K = G_v^{+_k}$ and $A \subseteq G^{+_k}$ as constructed in the proposition. We also know that G^{+_k} is either trivial or simple by Theorem 3.29. If G^{+_k} is trivial then G^{+_k} clearly also satisfies the contraction group property, so we may suppose that G^{+_k} is non-trivial and simple. By Theorem 4.7, we just need to show that the Cartan-like decomposition $G^{+_k} = \bigsqcup_{a \in A} KaK$ has the contraction group property.

Let $(a_i)_{i=1}^{\infty} \subseteq A$ be arbitrary; we need to show that $(a_i)_{i=1}^{\infty}$ is either bounded or has a subsequence with non-trivial contraction group. If the sequence is bounded, then we are done, so assume that the sequence is unbounded. Then we may assume, by passing to a subsequence if necessary, that for each $i \geqslant 1$ the distance from v to $a_i(v)$ is at least i and, since $\mathcal{T}$ is locally finite, that the first step of the path from v to $a_i(v)$ always passes through the same vertex, $w \in V\mathcal{T}$ say.

If infinitely many of the a_i are translations with v on the axis, by passing to a subsequence, we may suppose that they all are. Then choose $x \in G^{+_k}$ to fix $\mathcal{T}_{(w,v)}$ and act non-trivially on $\mathcal{T}_{(v,w)}$, which exists by the previous lemma. It is easy to check that $a_i x a_i^{-1}$ fixes the ball of radius i around v for each i and hence $a_i x a_i^{-1} \to \mathrm{id}$. So x is a non-trivial element in the contraction group of a subsequence of the a_i and we are done.

Similarly, if only finitely many of the a_i are translations with v on their axis, then it may be assumed that no a_i is a translation with v on its axis. Then each of the a_i are either elliptic elements or translations with v not on the axis. Also, for each i, w is closer than v to the fixed points of a_i, if a_i is elliptic, or the axis of a_i, if it is a translation. Choose $x \in G^{+_k}$ that fixes $\mathcal{T}_{(v,w)}$ and acts non-trivially on $\mathcal{T}_{(w,v)}$. It is easily checked that $a_i x a_i^{-1}$ fixes the ball of radius i around v for each i and hence converges to the identity. Then x is a non-trivial element of the contraction group of a subsequence of the a_i. This completes the proof. □

We now state a number of corollaries that result from this theorem:

Corollary 5.7. *Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ and suppose that G does not fix any proper non-empty subtree or fix an end of $\mathcal{T}$. Then $(G^{(k)})^{+_k}$ has the closed range property.*

PROOF. Since $G^{(k)}$ contains G, $G^{(k)}$ does not fix any non-empty subtree or end of $\mathcal{T}$. Also, $G^{(k)}$ is closed by Proposition 3.17. Now apply the previous theorem. □

Since the generalised universal groups $U_k(F)$ satisfy Property P_k and do not stabilise any proper non-empty subtree or fix an end of $\mathcal{T}$, this also gives us the following:

Corollary 5.8. *Let $F \leqslant \mathrm{Aut}(B_{d,k})$. Then $U_k(F)^{+_k}$ satisfies the closed range property.*

The theorem also allows us to show that the universal groups $U(F)$ satisfy the closed range property under mild assumptions:

Corollary 5.9. *Let $F \leqslant \mathrm{Sym}(d)$. Then $U(F)^+$ has the closed range property. Moreover, if F is transitive and generated by point stabilisers, then $U(F)$ has the closed range property.*

PROOF. That $U(F)^+$ has the closed range property is just a special case of the previous corollary. When F is transitive and generated by point stabilisers, $U(F)^+$ has index 2 in $U(F)$ by Theorem 3.5, and then an application of Proposition 4.10 shows that $U(F)$ has the closed range property. □

For use in the following corollary, a group $G \leqslant \mathrm{Aut}(\mathcal{T})$ is *locally semi-primitive* if for every $v \in V\mathcal{T}$, the vertex stabiliser G_v acts as a semi-primitive permutation group on the edges incident to v in $\mathcal{T}$. A permutation group is *semi-primitive* if it is transitive and all its normal subgroups are either transitive or free.

Corollary 5.10. *Let $G \leqslant \mathrm{Aut}(\mathcal{T})$ be closed, non-discrete and locally semi-primitive. If G does not fix any proper non-empty subtree or end of $\mathcal{T}$, and satisfies Property P_k, then G has the closed range property.*

PROOF. By the theorem, G^{+_k} has the closed range property, and by [**Tor20**, Proposition 2.11(iii)], G^{+_k} is cocompact in G since it is a normal subgroup of G. An application of Proposition 4.10 shows that G also has the closed range property. □

5.3. Commensurated Subgroups and the Closed Range Property

Let G be an arbitrary group. We say that two subgroups $H, K \leqslant G$ are *commensurated* if $[H : H \cap K] < \infty$ and $[K : K \cap H] < \infty$. Similarly, the subgroup H is said to be *commensurated* in G if $[H : gHg^{-1} \cap H] < \infty$ for all $g \in G$.

Commensurated subgroups are connected with the closed range property we are studying here. For example, the following result by Le Boudec–Wesolek in [**LBW19**] gives a link between commensurated subgroups and homomorphisms to totally disconnected locally compact groups having closed range:

Proposition 5.11. *Let G be a t.d.l.c. group such that every proper commensurated open subgroup of G is compact. Then every continuous homomorphism $\varphi : G \to H$ with H a t.d.l.c. group has closed range.*

In the paper [**LBW19**], Le Boudec and Wesolek also show that in almost automorphism groups of rooted trees, there are precisely three classes of closed commensurated subgroups:

THEOREM 5.12. [**LBW19**, Theorem 1.6] *If $\Lambda \leqslant \mathrm{AAut}(\mathcal{T}_{d,k})$ is commensurated, then either Λ is finite, $\overline{\Lambda}$ is compact and open, or $\Lambda = \mathrm{AAut}(\mathcal{T}_{d,k})$.*

As a result of this theorem and Proposition 5.3, the following closed range property for the almost automorphism groups is deduced:

Corollary 5.13. [**LBW19**, Corollary 7.1] *Every continuous homomorphism $\varphi : \mathrm{AAut}(\mathcal{T}_{d,k}) \to G$ with G a t.d.l.c. group has closed range.*

Proposition can also be used to prove the following result for discrete simple groups:

Proposition 5.14. *If G is a discrete simple group such that every proper commensurated subgroup is finite, and $\varphi : G \to H$ is a continuous homomorphism to a totally disconnected locally compact groups H, then φ has closed range.*

CHAPTER 6

Buildings and their Automorphism Groups

In this chapter we will give a brief introduction to the combinatorial approach to buildings and discuss some results concerning automorphism groups of right-angled buildings. We will also talk about some recent developments on universal groups of right-angled buildings, a generalisation of universal groups of regular trees. We start off by introducing some of the basic concepts involving the combinatorial approach to buildings and will more or less follow [**AB08**, Chapter 5]. It is assumed that the reader will already have some familiarity with Coxeter groups and the 'simplicial' approach to buildings.

Let S be a set and $M = (m(s,t))_{s,t\in S}$ be a square matrix indexed by the elements of S satisfying the following properties:

(i) $m(s,t) \in \mathbb{N} \cup \{\infty\}$ for all $s,t \in S$,
(ii) $m(s,s) = 1$ for all $s \in S$,
(iii) $2 \leqslant m(s,t) \leqslant \infty$ for all $s \neq t$,
(iv) $m(s,t) = m(t,s)$.

A matrix M satisfying the above properties is called a *Coxeter matrix.* We then define a group W_M given by the following presentation:

$$W_M := \langle S \mid (st)^{m(s,t)} = 1\rangle$$

and we interpret the relation $(st)^{m(s,t)} = 1$ when $m(s,t) = \infty$ to mean that there is no relation between the elements s and t. We often call W_M a *Coxeter group* with generating set S and refer to the pair (W_M, S) as a *Coxeter system.* Often we will drop the subscript M and merely denote a Coxeter system by (W,S) and interpret this to mean that W is a group with generating set S and presentation of the form above. The Coxeter system (W,S) is called *spherical* if W if finite.

For the remainder of this section, fix a Coxeter system (W, S) and let ℓ be the function that assigns to each word in W its length with respect to the generating set S. A *building* Δ of *type* (W, S) is a pair $(\mathrm{Ch}(\Delta), \delta)$, where $\mathrm{Ch}(\Delta)$ is a non-empty set whose elements are called the *chambers* of Δ, and $\delta : \mathrm{Ch}(\Delta) \times \mathrm{Ch}(\Delta) \to W$ a map called the *Weyl distance* function which satisfies the following properties:

(i) $\delta(C, D) = 1$ if and only if $C = D$.
(ii) If $\delta(C, D) = w$ and $C' \in \mathrm{Ch}(\Delta)$ satisfies $\delta(C', C) = s \in S$, then either $\delta(C', D) = w$ or $\delta(C', D) = sw$. If, in addition, $\ell(sw) = \ell(w) + 1$, then $\delta(C', D) = sw$.
(iii) If $\delta(C, D) = w$, then for any $s \in S$ there is a chamber $C' \in \mathrm{Ch}(\Delta)$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$.

We remark that it can be shown that the function δ satisfies $\delta(C, D) = \delta(D, C)^{-1}$ for any $C, D \in \mathrm{Ch}(\Delta)$. Further, δ satisfies the *gate property*, that is, $\delta(C, E) = \delta(C, D)\delta(D, E)$ for all $C, D, E \in \mathrm{Ch}(\Delta)$. These facts require proof which we will not give here, however they can be found in [**AB08**, Chapter 5] for instance. One will note that the properties of δ vaguely resemble the properties of a metric: the above definition of a building is also often referred to as a *W-Metric Space*.

Now, let $J \subseteq S$. Two chambers $C, D \in \mathrm{Ch}(\Delta)$ are said to be J-equivalent, which we denote by $C \sim_J D$, if $\delta(C, D) \in W_J$ where $W_J = \langle J \rangle \leqslant W$. It is straight forward to check that this is an equivalence relation on the set of chambers of Δ. The equivalence classes under this equivalence relation are called *J-residues*, and the J-residue containing the chamber $C \in \mathrm{Ch}(\Delta)$ will be denoted by $\mathcal{R}_J(C)$. An arbitrary subset $\mathcal{R} \subseteq \mathrm{Ch}(\Delta)$ is called a *residue* if it is a J-residue for some $J \subseteq S$. The set J is called the *type* of the residue and $|J|$ is called the *rank* (n.b. the rank of the building Δ is $|S|$.

If $J = \{s\}$ for some $s \in S$, we say that two chambers C and D are *s-equivalent* and write $C \sim_s D$. Moreover, if $\delta(C, D) = s$ then we say that C and D are *s-adjacent*, and two chambers are said to be *adjacent* if they are s-adjacent for some $s \in S$. The equivalence classes in $\mathrm{Ch}(\Delta)$ under the equivalence relation $\sim_s$ $(s \in S)$ are called *s-panels*. The term *panel* is used to refer to an s-panel for some $s \in S$. The unique s-panel containing a chamber $C \in \mathrm{Ch}(\Delta)$ will be denoted by $\mathcal{P}_s(C)$. A building such that every panel has cardinality two is called *thin*, and a building where every panel has cardinality strictly greater than two is called *thick*. A thin subbuilding of the building Δ is called an *apartment* of Δ.

A *gallery* of length n is a sequence of chambers $\Gamma : C_0, \ldots, C_n$ such that C_{i-1} is adjacent to C_i for each i. If there is no gallery of shorter length between C_0 and C_n, then we define the distance $d(C, D)$ between C and D to be n. One can show that $d(C, D) = \ell(\delta(C, D))$. The gallery Γ is called *minimal* if $d(C_0, C_n) = n$. The *type* of the gallery Γ is $\mathbf{s}(\Gamma) := (s_1, s_2, \ldots, s_n)$ where $s_i = \delta(C_{i-1}, C_i)$ for each i. It can also be checked that two chambers $C, D \in \mathrm{Ch}(\Delta)$ are in the same J-residue if and only if there is a gallery of type $(s_1, \ldots, s_n)$ connecting C to D such that $s_i \in J$ for each i.

Given a residue $\mathcal{R}$ and a chamber $D \in \mathrm{Ch}(\Delta)$, define $d(\mathcal{R}, D) := \min\{d(C, D) \mid C \in \mathrm{Ch}(\mathcal{R})\}$. It can be shown that there is a unique chamber $C_1 \in \mathrm{Ch}(\mathcal{R})$ such that $d(C_1, D) = d(\mathcal{R}, D)$ (c.f. [**AB08**, Proposition 5.34]). The chamber C_1 is then called the *projection* of D onto $\mathcal{R}$ and is denoted by $\mathrm{proj}_{\mathcal{R}}(D)$. For two residues $\mathcal{R}_1$ and $\mathcal{R}_2$, we define $\mathrm{proj}_{\mathcal{R}_1}(\mathcal{R}_2) := \{\mathrm{proj}_{\mathcal{R}_1}(C) \mid C \in \mathrm{Ch}(\mathcal{R}_2)\}$.

6.1. Right-Angled Buildings

A Coxeter system (W, S) is called right-angled if its Coxeter matrix $M = (m(s,t))_{s,t\in S}$ satisfies the property that $m(s, t) = 2$ or ∞ whenever $s \neq t$. A building Δ of type (W, S) is called right-angled if the Coxeter system (W, S) is a right-angled Coxeter system. An important result about right-angled buildings is the following result by Haglund-Paulin in [**HP03**]:

THEOREM 6.1. [**HP03**, Proposition 1.2] *Let (W, S) be a right-angled Coxeter system and $(q_s)_{s\in S}$ be a collection of cardinal numbers indexed by the elements of S such that $q_s \geqslant 2$ for each s. Then there exists a right-angled building of type (W, S) such that every s-panel has cardinality q_s. Moreover, this building is unique up to isomorphism.*

Such a building as described in the theorem where each s-panel has prescribed thickness q_s is called a *semi-regular* right-angled building. Next, we say that a group

G acts on a building Δ *strongly transitively* if G is transitive on pairs $(C, \mathcal{A})$ in Δ where C is a chamber and $\mathcal{A}$ is an apartment containing C. The following simplicity result was proved by Caprace in [**Cap14**] and will later be used to establish a closed range property for these groups:

Theorem 6.2. [**Cap14**, Theorem 1.1] *Let Δ be a thick semi-regular right-angled building of type (W, S). Assume that (W, S) is irreducible and non-spherical. Then the group $\mathrm{Aut}(\Delta)^+$ of type preserving automorphisms of Δ is abstractly simple and acts strongly transitively on Δ.*

If Δ is a semi-regular right-angled building with prescribed thicknesses $(q_s)_{s\in S}$ such that $q_s < \infty$ for each $s \in S$, then the automorphism group $\mathrm{Aut}(\Delta)$ is a compactly generated totally disconnected locally compact group with the permutation topology, and by the above theorem, the subgroup of type preserving automorphisms is a simple compactly generated totally disconnected locally compact group if (W, S) is non-spherical and irreducible.

Before moving on to understanding universal groups of right-angled buildings, we first need to define some more terminology useful for the study of right-angled buildings. First, given two panels $\mathcal{R}_1$ and $\mathcal{R}_2$, we say that $\mathcal{R}_1$ and $\mathcal{R}_2$ are *parallel* if $\mathrm{proj}_{\mathcal{R}_1}(\mathcal{R}_2) = \mathcal{R}_1$ and $\mathrm{proj}_{\mathcal{R}_2}(\mathcal{R}_1) = \mathcal{R}_2$. It can be shown that parallelism is an equivalence relation on the set of all residues. This is a corollary of the following statement from [**Cap14**]. For use in the following, for $J \subseteq S$, define $J^\perp = \{t \in S\backslash J \mid ts = st \text{ for all } s \in J\}$. When $J = \{s\}$, we use the notational convention that $J^\perp = s^\perp$.

Proposition 6.3. [**Cap14**, Proposition 2.8] *Let Δ be a right-angled building of type (W, S). The following properties hold:*

(i) Any two parallel residues have the same type.
(ii) Let $J \subseteq S$. Given a residue $\mathcal{R}$ of type J, a residue $\mathcal{R}'$ is parallel to $\mathcal{R}$ if and only if $\mathcal{R}'$ is of type J and $\mathcal{R}$ and $\mathcal{R}'$ are both contained in a common residue of type $J \cup J^\perp$.

An *s-tree-wall* is then defined as an equivalence class of parallel s-panels in Δ. For an s-tree-wall $\mathcal{T}$ we will let $\mathrm{Ch}(\mathcal{T})$ denote the set of all chambers of Δ contained in some panel of the equivalence class $\mathcal{T}$. As a result of the above proposition, the following is also true:

Proposition 6.4. [**DMSS18**, Corollary 2.25] *Let Δ be a right-angled building of type (W, S) and let $s \in S$. Two s-panels $\mathcal{P}_1$ and $\mathcal{P}_2$ belong to the same s-tree-wall if and only if they are both contained in a common residue of type $s \cup s^{\perp}$.*

6.2. Universal Groups for Right-Angled Buildings

In an analogous way to universal groups of regular trees, semi-regular right-angled buildings can be assigned a legal labelling and a notion of universal group can be defined. In this section we give a brief overview of the work from the paper [**DMSS18**] where universal groups of right-angled buildings were first defined. For the remainder of this section, we fix a semi-regular right-angled building Δ of type (W, S) and prescribed thicknesses $(q_s)_{s \in S}$.

First we define what a labelling of a semi-regular right-angled building is:

Definition 6.5 (s-Labelling)**.** For each $s \in S$, let Ω_s be a set of cardinality q_s, which is called the set of *s-labels*. A map $\lambda_s : \mathrm{Ch}(\Delta) \to \Omega_s$ is called an *s-labelling* of Δ if for every s-panel $\mathcal{P}$, there is a bijection between the chambers of $\mathcal{P}$ and the elements of Ω_s.

From here, one can then define a notion of 'legal-labelling' for semi-regular right-angled buildings:

Definition 6.6 (Legal s-Labelling)**.** An s-labelling $\lambda_s : \mathrm{Ch}(\Delta) \to \Omega_s$ is called a *legal s-labelling* if for every $S \backslash \{s\}$-residue $\mathcal{R}$, $\lambda_s(C) = \lambda_s(D)$ for all $C, D \in \mathrm{Ch}(\mathcal{R})$.

We remark that, given a legal s-labelling λ_s, every t-panel $\mathcal{P}$ for $t \in S \backslash \{s\}$ can be assigned a well defined label denoted $\lambda_s(\mathcal{P})$ since every chamber of $\mathcal{P}$ is assigned the same λ_s label. Another weaker notion of a legal-labelling is the following, which will be important in defining universal groups of semi-regular right-angled buildings:

Definition 6.7 (Weak Legal s-Labelling)**.** An s-labelling λ_s is called a *weak legal s-labelling* if whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are two s-panels in a common s-tree-wall, then for all $C \in \mathrm{Ch}(\mathcal{P}_1)$, we have $\lambda_s(C) = \lambda_s(\mathrm{proj}_{\mathcal{P}_2}(C))$.

Every legal s-labelling is indeed a weak legal s-labelling. It is also the case that every weak legal s-labelling is a legal s-labelling when restricted to $\mathrm{Ch}(\mathcal{T})$ for an s-tree-wall $\mathcal{T}$. Now, given two s-labelings λ_s and $\tilde{\lambda}_s$ and a group $G \leqslant \mathrm{Sym}(\Omega_s)$, we will say that the labelings λ_s and $\tilde{\lambda}_s$ are *G-equivalent* if for every s-panel $\mathcal{P}$, there is $g \in G$ such that $\lambda_s|_{\mathcal{P}} = g \circ \tilde{\lambda}_s|_{\mathcal{P}}$.

The following gives a relation between legal labelings and weak legal labelings:

Proposition 6.8. [**DMSS18**, Proposition 2.48] *Let $s \in S$ and $G \leqslant \mathrm{Sym}(\Omega_s)$ be a transitive permutation group. Then every weak legal s-labelling is G-equivalent to a legal s-labelling.*

We now come to the definition of a universal group of a semi-regular right-angled building:

Definition 6.9. Let Δ be a semi-regular right-angled building with prescribed thicknesses $(q_s)_{s\in S}$. For each $s \in S$, let $\lambda_s : \mathrm{Ch}(\Delta) \to \Omega_s$ be a weak legal s-labelling, where Ω_s is a set of cardinality q_s, and $G_s \leqslant \mathrm{Sym}(\Omega_s)$ a transitive permutation group. Define the universal group $U((G_s)_{s\in S})$ of Δ with respect to the groups $(G_s)_{s\in S}$ as:

$$U((G_s)_{s\in S}) = \{g \in \mathrm{Aut}(\Delta) \mid (\lambda_s|_{\mathcal{P}_s(gC)}) \circ g \circ (\lambda_s|_{\mathcal{P}_s(C)})^{-1} \in G_s, \text{ for all } s \in S,$$
$$\text{all } s\text{-panels } \mathcal{P}_s, \text{ and for all } C \in \mathcal{P}_s\}$$

The above definition of universal group does not depend on whether we start with a weak legal labelling or a non-weak legal labelling:

Proposition 6.10. [**DMSS18**, Lemma 3.2] *For each $s \in S$, let $(\lambda_s)_{s\in S}$ and $(\tilde{\lambda}_s)_{s\in S}$ be two G_s-equivalent labelings. Then the universal groups constructed using the labelings $(\lambda_s)_{s\in S}$ and $(\tilde{\lambda}_s)_{s\in S}$ coincide.*

Further, it is true that the definition of universal group does not depend on the choice of legal labelling; for different legal labelings the groups are conjugate to each other.

The *local action* of a group $H \leqslant \mathrm{Aut}(\Delta)$ at a panel $\mathcal{P}$ is defined to be the action of the set-wise stabiliser $H_{\mathcal{P}}$ on $\mathcal{P}$. The universal groups $U((G_s)_{s\in S})$ have the following property, analogous to the corresponding property for universal groups of regular trees:

Proposition 6.11. [**DMSS18**, Lemma 3.5] *The local action of the universal group on an s-panel is isomorphic to G_s for each $s \in S$.*

The universal groups $U(G_s)_{s\in S}$ also have the following universality property similar to universal groups of regular trees: given any closed chamber-transitive subgroup $H \leqslant \mathrm{Aut}(\Delta)$ such that the local action on each s-panel is isomorphic to G_s for each $s \in S$, then H is conjugate in $\mathrm{Aut}(\Delta)$ to a subgroup of $U((G_s)_{s\in S}$.

Universal groups of semi-regular right-angled buildings further share many similar properties to the universal groups of regular trees. The following proposition is extracted from Proposition 3.7 in [**DMSS18**]:

Proposition 6.12. *Let Δ be a semi-regular right-angled building with prescribed thicknesses $(q_s)_{s\in S}$. For each $s \in S$, let $G_s \leqslant \mathrm{Sym}(\Omega_s)$ be a finite transitive permutation group. Then the universal group $U((G_s)_{s\in S})$ satisfies the following properties:*

(i) $U((G_s)_{s\in S})$ *is a closed subgroup of* $\mathrm{Aut}(\Delta)$.
(ii) $U((G_s)_{s\in S})$ *is chamber transitive.*
(iii) If Δ is locally finite, then $U((G_s)_{s\in S})$ is compactly generated.

Proof. (i): The proof is much the same as the proof for universal groups of regular trees: we will show that $\mathrm{Aut}(\Delta)\backslash U((G_s)_{s\in S})$ is open. Let $g \in \mathrm{Aut}(\Delta)\backslash U((G_s)_{s\in S})$. Then there exists an s-panel $\mathcal{P}_s$ for some $s \in S$ and a chamber $C \in \mathrm{Ch}(\mathcal{P}_s)$ such that $(\lambda_s|_{\mathcal{P}_s(gC)}) \circ g \circ (\lambda_s|_{\mathcal{P}_s(C)})^{-1} \notin G_s$. The set of all automorphisms that agree with g on the panel $\mathcal{P}_s(C)$ is open in the permutation topology on $\mathrm{Aut}(\Delta)$ and is contained in $\mathrm{Aut}(\Delta)\backslash U((G_s)_{s\in S})$. Thus it folllows that $\mathrm{Aut}(\Delta)\backslash U((G_s)_{s\in S})$ is open and hence $U((G_s)_{s\in S})$ is closed in $\mathrm{Aut}(\Delta)$.

(ii): First suppose that C and D are two adjacent chambers in the building Δ. Then C and D are contained in a unique s-panel $\mathcal{P}$. The set-wise stabiliser $U((G_s)_{s\in S})_{\mathcal{P}}$ of the panel $\mathcal{P}$ is isomorphic to the group G_s. Since the group G_s is chamber transitive, it follows that there is an automorphism in $U((G_s)_{s\in S})_{\mathcal{P}}$ sending C to D. This shows that for any two adjacent chambers in Δ, there is an automorphism taking C to D. Then given any two arbitrary chambers C and D in Δ, find a minimal gallery $\Gamma : C = C_0, C_1, \ldots, C_n = D$ from C to D in Δ. By the previous arguments, for each i there exists an automorphism $g_i \in U((G_s)_{s\in S})$ sending C_i to C_{i+1} the composition of these automorphisms then sends C to D. This shows that $U((G_s)_{s\in S})$ is chamber transitive.

(iii): Fix a chamber $C \in \mathrm{Ch}(\Delta)$ and $C_1, \ldots, C_n \in \mathrm{Ch}(\Delta)$ be the chambers adjacent to C. For each $i \in \{1, \ldots, n\}$ choose $g_i \in U((G_s)_{s\in S}$ such that $g_i(C_i) = C$. We claim that the compact set $U((G_s)_{s\in S})_C \cup \{g_1, \ldots, g_n\}$ generates $U((G_s)_{s\in S}$. To do this, it suffices to show that for every $g \in U((G_s)_{s\in S}$, there exists $g' \in \langle g_1, \ldots, g_n\rangle$ such that $g'g(C) = C$. We prove this by induction on the distance from C to $g(C)$. If $d(C, g(C)) = 1$ then the result just follows by definition of the g_i. Now suppose that the result holds whenever $d(C, g(C)) \leqslant k$ and suppose that $d(C, g(C)) = k+1$. Find a minimal gallery $\Gamma : C, D_1, D_2, \ldots, D_{k+1} = g(C)$ between C and $g(C)$ in Δ. Then there exists a g_i such that $g_i(C) = D_1$. By the induction hypothesis, since $d(g_i(C), g(C)) = n$, there exists $g'' \in \langle g_1, \ldots, g_n\rangle$ such that $g''g_i(C) = C$. Then $g' = g''g_i$ is in $\langle g_1, \ldots, g_n\rangle$ and satisfies $g'g(C) = C$ which completes the proof. □

The universal groups of buildings are also (abstractly) simple under certain assumptions:

Theorem 6.13. [**DMSS18**, Theorem 3.25] *Let Δ be a thick right-angled building of irreducible type (W, S) with prescribed thicknesses $(q_s)_{s\in S}$ and rank at least 2. For each $s \in S$, let $\lambda_s : \mathrm{Ch}(\Delta) \to \Omega_s$ be a weak legal s-labelling and $G_s \leqslant \mathrm{Sym}(\Omega_s)$ a transitive permutation group generated by point stabilisers. Then the universal group $U((G_s)_{s\in S})$ is simple.*

CHAPTER 7

Cartan-like Decompositions of Automorphism Groups of Buildings

In this chapter, we study Cartan-like decompositions of automorphism groups of semi-regular right-angled buildings with the aim of initiating the study of the contraction group and closed range properties for these groups. We continue with a fixed semi-regular right-angled building Δ of type (W, S) and prescribed thicknesses $(q_s)_{s \in S}$. Further it is assumed that $q_s < \infty$ for each $s \in S$. We start out by proving that the group $\mathrm{Aut}(\Delta)^+$ admit a Cartan-like decomposition and the coset representatives can be chosen to be in one-to-one correspondence with the elements of W.

By Proposition 6.1 in [**Cap14**], the group $\mathrm{Aut}(\Delta)^+$ acts strongly transitively on Δ. For a fixed chamber $C \in \mathrm{Ch}(\Delta)$, the group of automorphisms in $\mathrm{Aut}(\Delta)^+$ that stabilise C, denoted $\mathrm{Aut}(\Delta)^+_C$, is a compact open subgroup of $\mathrm{Aut}(\Delta)^+$. The following proposition gives an enumeration of the coset representatives for a Cartan-like decomposition of $\mathrm{Aut}(\Delta)^+$:

Proposition 7.1. *Let Δ be a semi-regular right-angled building of type (W, S) with prescribed thicknesses $(q_s)_{s \in S}$ such that $q_s < \infty$ for each $s \in S$. Fix $C \in \mathrm{Ch}(\Delta)$ and let $K = \mathrm{Aut}(\Delta)^+_C$. The group $\mathrm{Aut}(\Delta)^+$ admits a Cartan-like decomposition $\mathrm{Aut}(\Delta)^+ = \bigsqcup_{a \in A} KaK$ for a collection of coset representatives $A \subseteq \mathrm{Aut}(\Delta)^+$. Moreover, A may be chosen to be in one-to-one correspondence with the elements of W.*

Proof. Fix an apartment $\mathcal{A}$ in Δ containing C. For each $w \in W$, choose a chamber $C' \in \mathrm{Ch}(\Delta)$ with Weyl distance w from C (i.e. $\delta(C, C') = w$) and choose an automorphism h_w mapping C to C'. Let A be the collection of all these h_w i.e. $A = \{h_w \mid w \in W\}$. We claim that $\mathrm{Aut}(\Delta)^+ = \bigsqcup_{a \in A} KaK$. Indeed, let $g \in \mathrm{Aut}(\Delta)^+$ and suppose that $\delta(C, g(C)) = w$. Choose an apartment $\mathcal{A}'$ containing C and $g(C)$. Then, by strong transitivity of $\mathrm{Aut}(\Delta)^+$, there exists an automorphism $k \in K$ mapping the pair $(C, \mathcal{A}')$ to the pair $(C, \mathcal{A})$. Since k is type-preserving, $\delta(C, kg(C)) = w$ and hence it follows that $h_w^{-1}kg(C) = C$. Thus there exists $k' \in K$ with $h_w^{-1}kg = k'$ i.e. $g = k^{-1}h_w k' \in \bigsqcup_{a \in A} KaK$. □

The next goal is to show that the above Cartan-like decomposition of $\mathrm{Aut}(\Delta)^+$ satisfies the contraction group property, however, the details of this proof have still not been fully worked out. The idea is to replicate the proof that the automorphism group of a label regular tree satisfies the contraction group property seen in [**CW20**] (and a similar argument was seen early in this article). In said proof, we are able to reduce the statement to showing that a sequence of coset representatives $(g_i)_{i \in I}$ has a non-trivial contraction group, where the g_i shift a fixed vertex v (in the case of buildings, this fixed vertex will be the chamber C in the above proposition) in the tree some arbitrary distance along an infinite path. We can make the same arguments for locally finite semi-regular right-angled buildings. In the case of trees, to find a non-trivial element in the contraction group of the g_i, we just need to choose a non-trivial element in the fixator of one of the semi-trees obtained by removing the edge $\{v, w\}$ from the tree, where w is the first vertex on the path along which the g_i shift the fixed vertex.

For right-angled buildings, we also get an analogue of semi-tree's called s-wings, and from a result of Caprace in [**Cap14**], the fixators of s-wings are non-trivial under light assumptions. It is hoped that we can use these facts to replicate the proof for trees, but as already mentioned, the details still need to be worked out. A successful proof that the above decomposition satisfies the contraction group property will then allow us to deduce closed range results for $\mathrm{Aut}(\Delta)^+$ and the universal groups $U(G_s)_{s \in S}$ under the assumptions required for simplicity.

CHAPTER 8

Conclusion

After giving the reader an overview of the various different examples of totally disconnected locally compact groups acting on trees, we have successfully studied the contraction group and closed range properties that arose in the paper [**CW20**] in greater detail, proving a number of results and giving a few examples that further our understanding of these properties. We transferred the closed range result given in *loc. cit.* to a larger class of totally disconnected locally compact groups acting on trees, which also includes a proof that the universal groups $U(F)$ satisfy the closed range property whenever F is transitive and generated by point stabilisers. The article finishes with us initiating the study of the contraction group and closed range properties for automorphism groups of buildings. This work furthers our understanding of totally disconnected locally compact groups and further illustrates some of the similarities that simple totally disconnected locally compact groups share with Lie groups, and more generally, connected locally compact groups.

Bibliography

[AB08] P. Abramenko and K. Brown, *Buildings: Theory and applications*, Graduate Texts in Mathematics, Springer-Verlag, 2008.

[BEW15] C. Banks, M. Elder, and G. A. Willis, *Simple groups of automorphisms of trees determined by their actions on finite subtrees*, Journal of Group Theory **18** (2015), no. 2, 235–261.

[BM00] M. Burger and S. Mozes, *Groups acting on trees: from local to global structure*, Publications Mathématiques de l'IHÉS **92** (2000), 113–150.

[Cap14] P. Caprace, *Automorphism groups of right-angled buildings: simplicity and local splittings*, Fundamenta Mathematicae **224** (2014), 17–51.

[CDM11] P. Caprace and T. De Medts, *Simple locally compact groups acting on trees and their germs of automorphisms*, Transform. Groups **16** (2011), 375–411.

[CM11] P. Caprace and N. Monod, *Decomposing locally compact groups into simple pieces*, Mathematical Proceedings of the Cambridge Philosophical Society **150** (2011), 97–128.

[CM18] P. Caprace and N. Monod (eds.), *New direction in locally compact groups*, London Mathematical Society Lecture Note Series, Cambridge University Press, 2018.

[CRW17a] P. Caprace, C. Reid, and G. Willis, *Locally normal subgroups of totally disconnected groups. part i: General theory*, Forum of Mathematics, Sigma **5** (2017), e11.

[CRW17b] ——, *Locally normal subgroups of totally disconnected groups. part ii: Compactly generated simple groups*, Forum of Mathematics, Sigma **5** (2017), e12.

[CW20] M. Carter and G. Willis, *Decomposition theorems for automorphism groups of trees*, Bulletin of the Australian Mathematical Society **92** (2020), 1–8.

[DMSS18] T. De Medts, A. Silva, and K. Struyve, *Universal groups for right-angled buildings*, Groups, Geometry and Dynamics **12** (2018), 231–287.

[Gle51] A. M. Gleason, *The structure of locally compact groups*, Duke Mathematics Journal **18** (1951), 85–104.

[Gle52] ——, *Groups without small subgroups*, Annales of Mathematics **56** (1952), 193–212.

[HP03] F. Haglund and F. Paulin, *Constructions arborescentes d'immeubles*, Math. Ann. **325** (2003), 137–164.

[Kap99] C. Kapoudjian, *Simplicity of neretin's group of spheromorphisms*, Ann. Inst. Fourier **49** (1999), 1225–1240.

[KM08] B. Krön and R.G. Möller, *Analogues of cayley graphs for topological groups*, Mathematische Zeitschrift **258** (2008), no. 637.

[LB16] A. Le Boudec, *Groups acting on trees with almost prescribed local action*, Comment. Math. Helv. **91** (2016), 253–293.

[LBW19] A. Le Boudec and P. Wesolek, *Commensurated subgroups in tree almost automorphism groups*, Groups, Geometry and Dynamics **13** (2019), 1–30.

[Led19] W. Lederle, *Coloured neretin groups*, Groups, Geometry and Dynamics **13** (2019), 467–510.

[MV12] R. Möller and J. Vonk, *Normal subgroups of groups acting on trees and automorphism groups of graphs*, Journal of Group Theory **15** (2012), 831–850.

[MZ52] D. Montgomery and L. Zippin, *Small subgroups of finite-dimensional groups*, Annales of Mathematics **56** (1952), 213–241.

[Ner84] Y. Neretin, *Unitary representations of the groups of diffeomorphisms of the p-adic projective line*, Funktsional. Anal. i Prilozhen. **18** (1984), 92–93.

[RZ10] L. Ribes and P. Zalesskii, *Profinite groups*, Springer-Verlag, 2010, Second Edition.

[Tit70] J. Tits, *Sur le groupe des automorphismes d'un arbre*, Essays on Topology and Related Topics **92** (1970), 188–211.

[Tor20] S. Tornier, *Groups acting on trees with prescribded local action*, arXiv:2002.09876 (2020).

[vD31] D. van Dantzig, *Studiën over topologische algebra*, Ph.D. thesis, Rijksuniversiteit Groningen, 1931.

[vD36] ———, *Zur topologischen algebra iii. brouwersche und cantorsche gruppen*, Compositio Mathematica **3** (1936), 408–426.

[Wil94] G. Willis, *The structure of totally disconnected locally compact groups*, Mathematische Annalen **300** (1994), 341–363.

[Wil99] J. Wilson, *Profinite groups*, Oxford University Press, 1999.